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Zitterbewegung and Quantum Jumps in Relativistic Schrödinger Theory

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Within the general framework of the relativistic Schrödinger theory, a new wave equation is identified which stands between Dirac's *four-component* spinor equation and the scalar *one-component* Klein–Gordon equation. It is a *two-component*, first-order wave equation in pseudo-Riemannian spacetime which on one hand can take account of the Zitterbewegung (similar to the Dirac theory), but on the other hand describes spinless particles (just like the Klein–Gordon theory). In this way it is demonstrated that spin and Zitterbewegung are independent phenomena despite the fact that both effects refer to a certain kind of internal motion. An extra variable for the internal motion can be introduced (similarly as in the Dirac theory) so that the new wave equation is reduced to the Klein–Gordon case when the internal variable takes its trivial value and the internal motion is not excited. The internal degree of freedom admits the occurence of quasi-pure states (i.e., a special subset of the mixtures), which undergo a transition to a pure state in finite time. If the initial configuration is already a pure state, this transition occurs in the form of a sudden jump to the final pure state. The coupling of the new wave field to gravity via the Einstein equations makes the Zitterbewegung manifest through the corresponding trembling of the extension of a Friedmann–Robertson–Walker universe.

1. INTRODUCTION

When the *Zitterbewegung* (i.e., trembling motion) of the relativistic electron was discovered in the early days of quantum theory [1], this phenomenon appeared as a kind of curiosity which even further complicated the difficult situation with the interpretation of the new theory. Many problems concerning the Zitterbewegung have since been settled and it is nowadays understood as a typical quantum effect of truly relativistic nature which is treated in any serious textbook about relativistic quantum mechanics (e.g.,

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ref. 2). In short, the relativistic nature of the Zitterbewegung is due to the fact that in Dirac's theory the electron wave function must be represented by a four-component spinor field which is transformed under the action of the group *Spin*(1,3) when a Lorentz transformation is applied to the coordinate basis of space-time. This kinematical necessity of dealing with a four-component wave function is then made plausible physically by saying that one needs two components for the particle states with its two spin orientations and two further components for the corresponding antiparticle states. The particle states are then endowed by the Dirac equation with a positive energy $E_+ \sim Mc^2$ and the antiparticle states with a negative energy $E_- \sim -Mc^2$ so that certain bilinear densities, e.g., the current density $j_\mu = \psi \cdot \gamma_\mu \cdot \psi$, receive a trembling component due to the difference frequency

$$\omega_z = \frac{E_+ - E_-}{\hbar} \sim \frac{2Mc^2}{\hbar} \tag{I.1}$$

Clearly, such a frequency will be observed for annihilation radiation, where the mass energy of an electron–positron pair is converted to electromagnetic radiation energy, but this energy ω_z cannot be observed during single-particle processes, e.g., for transitions between stationary states of the electron in the Coulomb field of a nucleus. Thus one may conclude that, for single-particle processes, the Zitterbewegung is some kind of artefact of the relativistic theory with no observable consequences. However, such a conclusion would be erroneous because the Zitterbewegung produces observable effects even for single-particle processes, e.g., the energy level shifts via the *Darwin term* [3]. Furthermore, the Zitterbewegung has been used for an intuitive deduction of the Lamb shift [4].

However, in view of the obvious physical relevance of the Zitterbewegung, it must appear somewhat strange that the relativistic nature of this phenomenon has not been studied more thoroughly in the literature; and it was not until recently that the Zitterbewegung received new interest and attention [5–7]. In fact, there are some unclarified questions which mainly refer to the circumstance that the Zitterbewegung was discovered in connection with the spin of the electron. On the other hand, it is always emphasized that the Zitterbewegung comes about through the *superposition* of the particle and antiparticle states, thus mixing up the relativistic and spin effects (actually we shall show that Zitterbewegung also occurs for *mixtures* and is endowed here with an even richer structure than for the pure states). But spin and relativity are two completely different things and are coupled merely incidentally in the Dirac theory via the homomorphism of the Lorentz group *SO*(1,3) and spin group *Spin*(1,3). Therefore, if it is true that the Zitterbewegung is a truly relativistic effect, then it should occur also for spinless (but relativistic) particles, e.g., for the scalar particles which are described by the Klein–

Gordon theory. However, such an important question as the occurrence of Zitterbewegung within the framework of the *Klein–Gordon* theory is scarcely studied in textbooks; sometimes one finds the side remark that the Foldy–Wouthuysen technique yields some expansion term which can be considered as the analogue of the Darwin term in the Dirac theory [3].

Thus we are not allowed to conclude that the Zitterbewegung can exist only in connection with the spin phenomenon. In order to avoid misunderstandings let us remark here that the Klein–Gordon theory also admits solutions of oscillatory character occuring with the Zitter frequency ω_z (mentioned above), but these motions are then of *external* character, in contrast to the *internal* nature of the trembling motions for the Dirac theory. Observe that, in this latter (internal) case, there always exist external variables (e.g., four-momentum p_μ) which do not take part in the trembling motion! But even if we restrict the notion of Zitterbewegung to the internal motions, it is not true that spin and Zitterbewegung must necessarily occur simultaneously. In order to become convinced of this assertion, it will be sufficient to present a new relativistic wave equation which stands "*between*" the Klein–Gordon equation (no spin) and the Dirac equation (spin plus Zitterbewegung) in the sense that this new equation is then capable of taking account of the *internal* Zitterbewegung of *spinless* particles. The point here is that the new wave equation is capable of describing particle–antiparticle *mixtures*, whereas the conventional Klein–Gordon theory can deal only with *superpositions* of particle–antiparticle states. The additional mixture degree of freedom can then be made responsible for the emergence of internal Zitterbewegung. The main aim of the present paper is to present this desired new wave equation, together with a thorough study of its Zitterbewegung properties. These latter properties may be tested conveniently by coupling the quantum matter to gravity in the well-known Einsteinian way so that the Zitterbewegung becomes physically evident by the corresponding trembling of the size of the universe.

The general technique for obtaining these results refers to the *relativistic Schrödinger theory* (RST), which provides us with a very general framework for all the relativistic wave equations [8–11]. Indeed, these latter equations turn out to be special *realizations* of RST according to which kind of typical fiber is applied for the construction of the vector bundle in which the wave functions $\psi(x)$ are living as the bundle sections. For instance, the four-component Dirac theory has been revealed as a $\mathbb{C}^4$-realization of RST [9, 11–13]; the Klein–Gordon–Higgs equations for an $SU(2)$ doublet are found to be a $\mathbb{C}^2$-realization of RST [10, 14]; the simplest case is the ordinary Klein–Gordon theory for a single scalar particle which is a $\mathbb{C}^1$-realization [15]. Clearly if one wants to treat a scalar two-particle system, one has to resort again to a $\mathbb{C}^2$-realization [16], which, however, differs from the $SU(2)$

doublet case [10, 14] by the choice of the gauge group [i.e., $U(1) \times U(1)$ for the two-particle system in place of $SU(2)$ for the doublet system]. The desired new wave equation turns out to be the $\mathbb{R}^2$-realization of RST. Observe here that the $\mathbb{R}^2$-realization is based upon the use of real two-component wave functions $\psi(x) = (\varphi_1(x), \varphi_2(x))$, whereas the $\mathbb{C}^1$-realization is based upon one-component complex wave functions $\psi(x) = \varphi_1(x) + i\varphi_2(x)$, but nevertheless both realizations are not equivalent! Rather, the $\mathbb{R}^2$-realization yields a much richer theory which embraces the $\mathbb{C}^1$-realization (i.e., the ordinary Klein–Gordon theory) as a special subcase, namely as the subset of the pure states. This point-particle subcase is just obtained by "freezing" the internal degree of freedom in the more general $\mathbb{R}^2$-realization.

The arrangement for presenting all these results is the following: First, we briefly sketch the notion of Zitterbewegung from two different viewpoints of the quantum formalism: *probabilistic* and *fluid-dynamic* (Section 2). Next, we generalize RST, which up to now has been written down only for complex-valued realizations. In this generalized version, RST can have both complex-valued and real-valued realizations, where the latter are then used in the present paper in order to study the Zitterbewegung (Section 3). Since we want to demonstrate a physical effect of the Zitterbewegung, we consider the dynamics of a Friedmann–Robertson–Walker universe which is filled with the trembling quantum matter. The geometric prerequisites for such a model are collected in Section 4. Then the $\mathbb{C}^1$-realization of RST (i.e., Klein–Gordon theory) is studied in detail in order to reveal its special character within the larger formalism. These results are used later to demonstrate the absence of *internal* motions in the ordinary Klein–Gordon theory (Section 5). Finally, the $\mathbb{R}^2$-realization is studied in great detail with respect to the emergence of the Zitterbewegung (Section 6). Since the $\mathbb{R}^2$-realization does account for the Zitterbewegung (both internal and external), but not for the spin, it is thus demonstrated that these two phenomena of spin and internal Zitterbewegung are completely independent things. In addition to the existence of Zitterbewegung with its oscillatory character, the $\mathbb{R}^2$-realization predicts also the occurence of *nonoscillatory*, jumplike transitions from one pure state to another one (Section 7). But the intermediate field configurations are themselves not pure states, but a special type of mixture, the "quasi-pure" states. Such a behavior is not possible for a strictly unitary time evolution (as encountered in conventional quantum theory when using a Hermitian Hamiltonian).

2. ZITTERBEWEGUNG

From the many debates about the right interpretation of the quantum mechanical formalism [17], one should have learnt that a mathematical formalism generally admits more than one physical interpretation. Nowadays, most physicists would adhere to Born's *probabilistic* interpretation of the quantum formalism, which, however, seems to suffer from certain difficulties, too (see different views of the measurement problem in ref. 18). In contrast, the relativistic Schrödinger theory is a *fluid-dynamic* approach to the quantum world; but since both approaches rely upon the same mathematical formalism (i.e., Hilbert spaces, wave functions, operators, etc.), or even upon the same wave equations for one-particle systems (e.g., Klein–Gordon, Dirac), every feature of the formal apparatus can be interpreted from either of the two different viewpoints. Therefore let us briefly sketch the one-particle Zitterbewegung with respect to both interpretations.

2.1. Probabilistic Approach

Today we know that the Zitterbewegung is a truly relativistic effect, but, paradoxically, Schrödinger [1] discovered this phenomenon not by considering Dirac's equation in its manifestly invariant form

$$i\hbar\gamma^{\mu}\mathcal{D}_{\mu}\psi = Mc\psi \tag{2.1}$$

but by recasting this into the nonrelativistic form

$$i\hbar\,\frac{\partial\psi}{\partial t} = \hat{H}\cdot\psi \tag{2.2}$$

which he himself had proposed previously. Probably, he preferred such a procedure because one could then describe the particle by means of the well-known point-particle concepts of nonrelativistic mechanics such as position $\vec{x}$, momentum $\vec{p}$, energy E, etc., albeit in operator form ($\hat{\vec{x}}$, $\hat{\vec{p}}$, $\hat{H}$, say). The corresponding observable quantities could simply be obtained as an ensemble average with respect to the initial physical state $|\psi(0)\rangle$, e.g., for position

$$\overline{\vec{x}}(t) = \langle\psi(0)|\hat{\vec{x}}|\psi(0)\rangle \tag{2.3}$$

where the operators moved in the Heisenberg picture according to

$$\frac{d\hat{\vec{x}}(t)}{dt} = \frac{i}{\hbar}\,[\hat{H}, \hat{\vec{x}}(t)] \tag{2.4}$$

But surprisingly enough, when using the Dirac Hamiltonian for a free particle

$$\hat{H} = c\vec{\alpha}\cdot\vec{p} + \beta\cdot Mc^2 \tag{2.5}$$

Schrödinger did not find from (2.4) the velocity $\dot{\vec{x}}$ being proportional to momentum $\vec{p}$, but instead he was forced to identify the velocity with the (4×4) matrix $\vec{\alpha}$,

$$\frac{d\hat{\vec{x}}}{dt} = c\vec{\alpha} \tag{2.6}$$

This result was a great surprise because the momentum turned out to be a true constant of the motion,

$$\frac{d\hat{\vec{p}}(t)}{dt} = \frac{i}{\hbar}\,[\hat{H}, \hat{\vec{p}}(t)] \equiv 0 \tag{2.7}$$

as must be expected for a free particle. The generally accepted interpretation [2] of these curious results is based upon the formal solution $\hat{\vec{x}}(t)$ of Eq. (2.6):

$$\hat{\vec{x}}(t) = \hat{\vec{x}}(0) + c^2\hat{H}^{-1} \cdot \hat{\vec{p}} \cdot t + i\,\frac{\hbar c}{2\hat{H}}\exp\left(\frac{-2i\hat{H}t}{\hbar}\right) \cdot \tilde{\vec{\alpha}}(0) \tag{2.8}$$

Here the uniform motion ($\sim t$) of the particle is superposed by an oscillatory component of the Zitter frequency ω_z ($\sim 2Mc^2/\hbar$) and of amplitude $\hbar/2Mc$, i.e., the Compton length. Thus, these results seem to suggest that it is only the average path of the free particle which is a straight line (*external motion*), but the actual path resembles more some kind of trembling within a narrow tube of Compton width (*internal motion*) around that average path; see the illustrative drawing in ref. 2.

Clearly, it is very tempting now to think that the spin is a kind of residual regularity of the rather irregular trembling motion. Indeed, it is easily verified that the orbital angular momentum $\vec{L}$ itself is not a constant of the motion,

$$\frac{d\hat{\vec{L}}(t)}{dt} = \frac{i}{\hbar}\,[\hat{H}, \hat{\vec{L}}(t)] \neq 0 \tag{2.9}$$

but only the total angular momentum $\vec{J}$:

$$\hat{\vec{J}} \doteqdot \hat{\vec{L}} + \hat{\vec{S}} \tag{2.10a}$$

$$\hat{\vec{S}} \doteqdot -\frac{i\hbar}{4}\,(\vec{\alpha} \times \vec{\alpha}) \tag{2.10b}$$

i.e., we have the angular momentum conservation law

$$\frac{d\hat{\vec{J}}(t)}{dt} = \frac{i}{\hbar}\,[\hat{H}, \hat{\vec{J}}(t)] \equiv 0 \tag{2.11}$$

The equation of motion for the spin $\hat{\vec{S}}(t)$ is found as

$$\frac{d\hat{\vec{S}}(t)}{dt} = \frac{i}{\hbar}\,[\hat{H}, \hat{\vec{S}}(t)] = c\vec{\alpha} \times \hat{\vec{p}} \tag{2.12}$$

so that the projection of the spin onto the momentum $\vec{p}$ remains constant in time. The solution $\vec{S}(t)$ is found to have a Zitter component, too:

$$\hat{\vec{S}}(t) = \hat{\vec{S}}(0) + \hat{\vec{S}}(0) \exp\left(\frac{-2i\hat{H}t}{\hbar}\right) \cdot \frac{\hbar}{2\hat{H}} \tag{2.13}$$

which seems to confirm the supposition that spin and Zitterbewegung essentially might be the same thing. However, considering this question now from the fluid-dynamic point of view for other wave equations, will demonstrate the independency of both phenomena. As preparation let us first keep for a moment to the Dirac theory in order to inspect the specific way in which the Zitterbewegung arises from the *fluid-dynamic* point of view.

2.2. Fluid-Dynamic Approach

Naturally, the observable quantities for the fluid-dynamic approach cannot coincide with those of point-particle mechanics (classical or quantum mechanical) which have been used for the preceding probabilistic approach. Rather, one has to think now in terms of *physical densities*, which, however, are again constructed by Hermitian operators and the wave functions $\psi(x)$, e.g., for the current density $j_\mu(x)$ in Dirac's theory

$$^{(D)}j_\mu(x) \doteqdot \overline{\psi}(x) \cdot \gamma_\mu \cdot \psi(x) \tag{2.14}$$

Here the Dirac matrices $\{\gamma_\mu\}$ play the part of a velocity operator [11]. Nevertheless in the fluid-dynamic approach, too, there arises the phenomenon of Zitterbewegung as an internal motion quite analogously as in the probabilistic approach. One could even say that these phenomena can be treated much more rigorously and explicitly in fluid-dynamic terms because one can introduce here extra dynamical variables for the internal degree of freedom in a very natural way.

For instance, it has been demonstrated that the Dirac current j_μ, (2.14), can be reparametrized in terms of an orthonormal tetrad $\{b_\mu; \tilde{g}_\mu, \tilde{\pi}_\mu, \tilde{\lambda}_\mu\}$ and of scalar fields $\{\rho, \kappa, \chi, z\}$ in such a way that the timelike flow vector b_μ and the scalar density ρ $(=\overline{\psi} \cdot \psi)$ describe the *external* motion, whereas the *internal* motion is characterized by the spacelike triad $\{\tilde{g}_\mu, \tilde{\pi}_\mu, \tilde{\lambda}_\mu\}$ and by the remaining scalar fields $\{\kappa, \chi, z\}$:

$$\begin{aligned} ^{(D)}j_\mu = \rho(b_\mu \cdot \cosh 2\kappa + (\tilde{g}_\mu \cdot \cos\chi \\ + \tilde{\pi}_\mu(1 - z^2)^{1/2} \cdot \sin\chi) \cdot \sinh 2\kappa) \end{aligned} \tag{2.15}$$

[12, 13, 19]. The "strength" of excitation of the internal degree of freedom is described by the scalar κ, so that for $\kappa \Rightarrow 0$ the current j_μ is reduced to its purely *convective* constituent, i.e., we are left with the external motion:

$$^{(D)}j_\mu \Rightarrow \rho \cdot b_\mu \tag{2.16}$$

However, for $|\kappa| > 0$ the internal motion is excited and the current j_μ, (2.15), develops a trembling mode in the 2-plane spanned by the 2-frame $\{\tilde{g}_\mu, \tilde{\pi}_\mu\}$. Obviously, this is the fluid-dynamic counterpart of the Zitterbewegung (2.8) in the point-particle picture. This supposition is readily confirmed by an explicit computation of the space-time behavior of the trembling phase χ for a free particle [13]:

$$\chi \Rightarrow \frac{2Mc}{\hbar}\, b_\mu x^\mu \tag{2.17}$$

This nice result says nothing else than that the previous Zitter frequency ω_z sets the time scale for this trembling motion:

$$\dot{\chi} \doteqdot b^\mu\, \partial_\mu \chi = \frac{2Mc}{\hbar} \equiv \frac{\omega_z}{c} \tag{2.18}$$

Cf. (1.1).

But the essential point is now that one is not allowed to conclude that this Zitterbewegung can occur only for spinning particles. Actually, in Section 6 we will elaborate the Zitterbewegung for the current j_μ of a *scalar* particle and we will see that also in that case the current j_μ can be split into two parts. In Dirac's theory of spinning particles, this splitting refers to the emergence of a convection part ${}^{(C)}j_\mu$ and a polarization part ${}^{(P)}j_\mu$:

$${}^{(D)}j_\mu = {}^{(C)}j_\mu + {}^{(P)}j_\mu \tag{2.19}$$

where

$${}^{(C)}j_\mu = \rho \cdot b_\mu \cdot \cosh 2\kappa \tag{2.20a}$$

$${}^{(P)}j_\mu = \rho \cdot (\tilde{g}_\mu \cdot \cos \chi + \tilde{\pi}_\mu (1 - z^2)^{1/2} \cdot \sin \chi) \cdot \sinh \kappa \tag{2.20b}$$

See the discussion of the Dirac case in ref. 11.

Besides the current density j_μ, there are further physical densities of relevance with respect to Zitterbewegung. Let us mention here only the energy-momentum density $T_{\mu\nu}$ carried by the wave function ψ. This object $T_{\mu\nu}$ becomes especially simple for the special situation when the quantum fluid is distributed homogeneously and isotropically over a Friedmann–Robertson–Walker universe; cf. (4.3) below. The reason is that only the energy density U and pressure P enter this energy-momentum density $T_{\mu\nu}$ and therefore these two densities U and P are sufficient to determine the expansion dynamics of the universe according to Einstein's equations. But clearly, when U and P are infected by the Zitterbewegung, then the "radius" $\mathcal{R}$ of the universe will react also by developing a trembling mode and exactly this has been observed by means of numerical solutions for the coupled Dirac–Einstein equations [19–21]:

$$U \Rightarrow U_D = 3\hbar c\rho\left(\frac{Mc}{3\hbar} + \frac{\cos\chi}{2\mathcal{R}}\right) \tag{2.21a}$$

$$P \Rightarrow P_D = \hbar c\rho\,\frac{\cos\chi}{2\mathcal{R}} \tag{2.21b}$$

Here, the trembling variable χ obeys the equation of motion

$$\dot{\chi} \doteqdot b^{\mu}\,\partial_{\mu}\chi = \frac{Mc}{\hbar}\left(2 + 3\,\frac{\cos\chi}{\mathcal{R}}\right) \tag{2.22}$$

which explicitly displays the gravitational effect being superimposed over the free-particle case, (2.18). Thus both densities U and P are actually trembling roughly with the Zitter frequency ω_z and this prevents the universe from collapsing. The interesting point is here that this same effect also is obtainable without resorting to spinning matter, namely by evoking the $\mathbb{R}^2$-realization of RST; see the results in Section 7.

3. GENERALIZING RST

The discussion of the analogous trembling effects within the framework of RST is facilitated considerably by recasting this theory into a more general form so that its potential of covering both real and complex realizations can be better elucidated. Originally, the RSE was set up in the form [8–10]

$$i\hbar c\mathcal{D}_{\mu}\psi = \mathcal{H}_{\mu}\cdot\psi \tag{3.1}$$

on account of its formal analogy to the nonrelativistic Schrödinger equation

$$i\hbar\,\frac{\partial\psi}{\partial t} = \hat{H}\cdot\psi \tag{3.2}$$

But by this procedure it is unnecessarily suggested that the wave function ψ should always be an element of a *complex* space, which, however, is too restrictive. The motivation for adopting Eq. (3.1) as the true relativistic generalization of the nonrelativistic conventional form (3.2) was based upon the conviction that the motion of relativistic matter should also be organized by some kind of “Hamiltonian” ($\mathcal{H}_{\mu}$) which on the one hand respects the gauge nature of the interactions of matter and on the other hand guarantees the validity of certain conservation laws (e.g., for charge, energy-momentum, etc.). As a consequence, the relativistic Hamiltonian $\mathcal{H}_{\mu}$, (3.1), in contrast to its nonrelativistic counterpart $\hat{H}$, (3.2), had itself to be considered as a dynamical object of the theory and therefore must first be determined from its field equations (see below). But the crucial point now is that the RSE

(3.1) is meaningful only if the Hamiltonian $\mathcal{H}_\mu$ is taken to be *non-Hermitian* ($\overline{\mathcal{H}}_\mu \neq \mathcal{H}_\mu$). Therefore it may retrospectively appear somewhat artificial to include the imaginary unit (i) into the RSE and it seems more adequate to rewrite that equation simply as

$$\hbar c \mathcal{D}_\mu \psi = \mathcal{H}_\mu \cdot \psi \tag{3.3}$$

If one wants, one can restore the old form (3.1) from the new one (3.3) by means of the replacement $\mathcal{H}_\mu \to -i\mathcal{H}_\mu$, but the latter form (3.3) appears as the more general one because it suggests we consider also real-valued *realizations* of RST. Indeed, if one starts with the old form (3.1), one feels automatically forced to think of the Hamiltonian $\mathcal{H}_\mu$ ($\neq \overline{\mathcal{H}}_\mu$) as a $\mathfrak{gl}(N, \mathbb{C})$-valued 1-form, whereas the new form (3.3) *additionally* admits also $\mathfrak{gl}(N, \mathbb{R})$-valued Hamiltonian 1-forms. Subsequently, we will compare the complex 1-dimensional realization [$\Rightarrow \mathcal{H}_\mu \in \mathfrak{gl}(1, \mathbb{C})$] to the real 2-dimensional realization [$\Rightarrow \mathcal{H}_\mu \in \mathfrak{gl}(2, \mathbb{R})$] and we will benefit by the inequivalence of both realizations. As a preparation, let us first reformulate the RST from a generalized point of view.

3.1. Conservation Laws

As mentioned above, the point with RST is that it represents a rather general framework for relativistic fields which is required to guarantee only a few characteristics of the motion of matter (i.e., conservation laws), but otherwise admits a large variety of field configurations. The most relevant of the conservation laws are the following:

(i) *The charge conservation laws*:

$$D^\mu j_{a\mu} = 0 \tag{3.4}$$

$$(D_\mu j_{a\nu} \doteqdot \nabla_\mu j_{a\nu} - j_{b\nu}\omega^b{}_{a\mu})$$

Here, the gauge currents $j_{a\mu}$ are members of a gauge multiplet transforming as usual with respect to some change of gauge ($S^b{}_a$):

$$j'_{a\mu} = j_{b\mu} S^b{}_a \tag{3.5}$$

The representation $\mathbf{S} = \{S^b{}_a\}$ of the gauge group and the corresponding bundle connection $\omega^b{}_{a\mu}$ are N_p-fold reducible for a system of N_p particles [22].

(ii) *The energy-momentum conservation law for* closed *systems*:

$$\nabla^\mu T_{\mu\nu} = 0 \tag{3.6}$$

For the *nonclosed* matter subsystem, the source of the energy-momentum density $T_{\mu\nu}$ is the Lorentz force density f_ν,

$$\nabla^{\mu} T_{\mu\nu} = f_{\nu} \tag{3.7}$$

where f_{ν} should be composed linearly of both the current densities $j_{a\mu}$ and of the field strength $\mathcal{F}_{\mu\nu}$ (= bundle curvature) which is generated by the gauge potential $\mathcal{A}_{\mu}$ (= bundle connection) in the usual way,

$$\mathcal{F}_{\mu\nu} = \nabla_{\mu}\mathcal{A}_{\nu} - \nabla_{\nu}\mathcal{A}_{\mu} + [\mathcal{A}_{\mu}, \mathcal{A}_{\nu}] \tag{3.8}$$

Now in order to construct a matter field theory obeying all these requirements, one can write down the relativistic von Neumann equation for the intensity matrix $\mathcal{I}$ [as the mixture generalization of the pure states ψ, (3.3)]:

$$\mathcal{D}_{\mu}\mathcal{I} = \frac{1}{\hbar c} (\mathcal{H}_{\mu} \cdot \mathcal{I} + \mathcal{I} \cdot \overline{\mathcal{H}}_{\mu}) \tag{3.9}$$

and then one can construct the physical densities (e.g., $j_{a\mu}$ and $T_{\mu\nu}$) by means of the following recipe:

$$j_{a\mu} = \mathrm{tr}(\mathcal{I} \cdot v_{a\mu}) \tag{3.10a}$$

$$T_{\mu\nu} = \mathrm{tr}(\mathcal{I} \cdot \mathcal{T}_{\mu\nu}) \tag{3.10b}$$

Here, the physical densities are obviously obtained by forming the trace of the operator product of the intensity matrix $\mathcal{I}$ and the corresponding (Hermitian) "*observable*," e.g., the gauge velocity operators $v_{a\mu}$ (= $\bar{v}_{a\mu}$) or the energy-momentum operator $\mathcal{T}_{\mu\nu}$ (= $\overline{\mathcal{T}}_{\mu\nu}$). The physical densities fall into two subclasses, *internal* densities, which are *gauge covariant* (e.g., $j_{a\mu}$), and *external* densities, which are *gauge invariant* and therefore may be considered as truly observable quantities of the theory. Consequently, one will expect that the total motion of the matter subsystem is composed of an internal part and an external part, where the internal part is governed by the gauge nature of the theory. Subsequently we shall demonstrate that this internal motion is the kinematical origin of the *Zitterbewegung*.

But clearly, the kinematical ansatz (3.10) for the conserved quantities is not sufficient to ensure the desired conservation laws (3.4) and (3.6)–(3.7). Additionally, one must subject the observables to some kind of *conservation equations*, e.g., for the velocity operators $v_{a\mu}$

$$\mathcal{D}^{\mu} v_{a\mu} - v_{b\mu}\omega^{b}{}_{a}{}^{\mu} = -\frac{1}{\hbar c} (v_{a}{}^{\mu} \cdot \mathcal{H}_{\mu} + \overline{\mathcal{H}}_{\mu} \cdot v_{a}{}^{\mu}) \tag{3.11}$$

or similarly for the energy-momentum operator $\mathcal{T}_{\mu\nu}$,

$$\mathcal{D}^{\mu}\mathcal{T}_{\mu\nu} + \frac{1}{\hbar c} (\overline{\mathcal{H}}^{\mu} \cdot \mathcal{T}_{\mu\nu} + \mathcal{T}_{\mu\nu} \cdot \mathcal{H}^{\mu}) = \mathcal{F}_{\nu} \tag{3.12}$$

where the force operator $\mathcal{F}_{\nu}$ generates the force density f_{ν}, (3.7), in the usual way,

$$f_\nu = \mathrm{tr}(\mathcal{J} \cdot \mathcal{F}_\nu) \tag{3.13}$$

Observe here again that the conservation equation for an internal object, e.g., for $v_{a\mu}$, (3.11), requires the specification of a corresponding representation $\{\omega^b{}_{a\mu}\}$ of the gauge potential $\mathcal{A}_\mu$. This, however, can easily be obtained by specification of the (anti-Hermitian) generators τ_a $(= -\bar{\tau}_a)$ of the gauge group. More concretely, denote the structure constants of the gauge group by $C^{ab}{}_c$:

$$[\tau^a, \tau^b] = C^{ab}{}_c\, \tau^c \tag{3.14}$$

and define the corresponding fiber metric g^{ab} for raising and lowering fiber indices through

$$g^{ab} = -\mathrm{tr}(\tau^a \cdot \tau^b) \tag{3.15}$$

Then the desired representation $\{\omega^b{}_{a\mu}\}$ of the gauge potential $\mathcal{A}_\mu$ is given simply by the covariant derivative of the generators τ^a, i.e.,

$$\mathcal{D}_\mu \tau_a = \tau_b\, \omega^b{}_{a\mu} \tag{3.16}$$

$$(\mathcal{D}_\mu \tau_a \doteqdot \partial_\mu \tau_a + [\mathcal{A}_\mu, \tau_a])$$

Or, more explicitly, by means of the decomposition of $\mathcal{A}_\mu$,

$$\mathcal{A}_\mu = A_{a\mu} \cdot \tau^a \tag{3.17}$$

we find the connection coefficients $\omega^b{}_{a\mu}$ as

$$\omega^b{}_{a\mu} = A_{c\mu} C^c{}_a{}^b - \mathrm{tr}(\tau^b \cdot \partial_\mu\, \tau_a) \tag{3.18}$$

As an implication of this construction, the fiber metric g_{ab}, (3.15), is covariantly constant:

$$D_\mu g_{ab} \equiv 0 \tag{3.19}$$

$$(D_\mu g_{ab} \doteqdot \partial_\mu g_{ab} - g_{cb}\omega^c{}_{a\mu} - g_{ac}\omega^c{}_{b\mu})$$

Subsequently, we shall treat the 1-dimensional Abelian groups $U(1)$ and $SO(2)$ as simple examples for the gauge group, and correspondingly we expect (quasi-) periodic Zitterbewegung on account of the compact topology of these groups.

But now we have to face the problem of how to determine the velocity operators $v_{a\mu}$ and the energy-momentum operator $\mathcal{T}_{\mu\nu}$ in order that those conservation equations (3.11) and (3.12) can be satisfied. This problem has been dealt with in preceding papers, e.g., in ref. 10, and therefore it is sufficient here to merely quote the results, adapted to the present generalization of RST:

$$v_{a\mu} = -\frac{1}{2Mc^2}\,(\tau_a \cdot \mathcal{H}_\mu - \overline{\mathcal{H}}_\mu \cdot \tau_a) \tag{3.20}$$

$$\mathcal{T}_{\mu\nu} = \frac{1}{2Mc^2}\,(\overline{\mathcal{H}}_\mu \cdot \mathcal{H}_\nu + \overline{\mathcal{H}}_\nu \cdot \mathcal{H}_\mu - g_{\mu\nu}(\overline{\mathcal{H}}^\lambda \cdot \mathcal{H}_\lambda - (\mathcal{M}c^2)^2)) \tag{3.21}$$

Here $\mathcal{M}$ is the (Hermitian) mass operator which we consider covariantly constant

$$\mathcal{D}_\mu \mathcal{M} \equiv 0 \tag{3.22}$$

so that its trace M

$$M = \frac{1}{N_f}\,\mathrm{tr}\,\mathcal{M} \tag{3.23}$$

is a constant mass parameter of the theory ($N_f = \mathrm{tr}\,\mathbf{1}$ is the fiber dimension for the system of N_f particles). However, the propositions (3.20)–(3.21) are not yet the ultimate solution of the problem because the desired operators $v_{a\mu}$ and $\mathcal{T}_{\mu\nu}$ have been expressed merely in terms of the Hamiltonian $\mathcal{H}_\mu$ and this alone does not yet guarantee the validity of the conservation equations (3.11) and (3.12). Obviously it becomes necessary to specify a suitable conservation equation for the Hamiltonian $\mathcal{H}_\mu$ itself [8, 10]:

$$\mathcal{D}^\mu \mathcal{H}_\mu + \frac{1}{\hbar c}\,\mathcal{H}^\mu \mathcal{H}_\mu = -\hbar c\left(\frac{\mathcal{M}c}{\hbar}\right)^2 \tag{3.24}$$

Indeed one can easily show that both the conservation equations for $v_{a\mu}$, (3.11), and $\mathcal{T}_{\mu\nu}$, (3.12), are satisfied on account of the present postulate (3.24), provided the mass operator is a gauge *invariant*:

$$[\mathcal{M}, \tau_a] = 0, \qquad \forall a \tag{3.25}$$

Furthermore, the force operator acquires the expected Lorentz-type form:

$$\mathcal{F}_\nu = \frac{\hbar}{2Mc}\,[\overline{\mathcal{H}}^\mu \cdot \mathcal{F}_{\mu\nu} - \mathcal{F}_{\mu\nu} \cdot \mathcal{H}^\mu] \tag{3.26}$$

The well-known Lorentz force density f_ν, (3.13), is then immediately obtained from this result by decomposing the field strength $\mathcal{F}_{\mu\nu}$ in a similar way to its potential $\mathcal{A}_\mu$, (3.17):

$$\mathcal{F}_{\mu\nu} = F_{a\mu\nu}\tau^a \tag{3.27}$$

This then yields for the force operator (3.26) in terms of the velocity operators

$$\mathcal{F}_\nu = \hbar c F_{a\mu\nu} \cdot v^{a\mu} \tag{3.28}$$

which is nothing else than the operator version of the old Lorentzian idea that the force should be the product of field strength and velocity. Finally the force density f_ν, (3.13), is also found to be of the expected form

$$f_\nu = \hbar c F_{a\mu\nu} \cdot j^{a\mu} \tag{3.29}$$

where the previous definition (3.10a) for the current densities $j_{a\mu}$ has been applied.

3.2. Integrability Condition

Now that we can be sure that both conservation laws (3.4) and (3.6)–(3.7) are safely valid in our theory, we can turn to the question of the existence of solutions, both to the RSE (3.3) and to the relativistic von Neumann equation (3.9). This leads us to the question of integrability conditions. For instance, in order to be sure that a gauge potential $\mathcal{A}_\mu$ really exists for the field strength $\mathcal{F}_{\mu\nu}$, (3.8), the well-known Bianchi identity must be satisfied:

$$\mathcal{D}_\lambda \mathcal{F}_{\mu\nu} + \mathcal{D}_\mu \mathcal{F}_{\nu\lambda} + \mathcal{D}_\nu \mathcal{F}_{\lambda\mu} \equiv 0 \tag{3.30}$$

Or, in order that the solutions $\psi(x)$ to the RSE (3.3) are proper sections of the corresponding vector bundle, they must obey the bundle identity

$$[\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu]\psi \equiv \mathcal{F}_{\mu\nu} \cdot \psi \tag{3.31}$$

Similarly, for the intensity matrix $\mathcal{I}(x)$ as a solution to the von Neumann equation (3.9) we must have

$$[\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu]\mathcal{I} \equiv [\mathcal{F}_{\mu\nu}, \mathcal{I}] \tag{3.32}$$

Consequently, the problem is now to construct the dynamics for the Hamiltonian $\mathcal{H}_\mu$ in such a way that all these identities are automatically obeyed (by the corresponding solutions of the theory). As was shown in preceding papers [8, 10], this can be attained by completing the conservation equation for $\mathcal{H}_\mu$, (3.24) by an additional curl relation, the *integrability condition*:

$$\mathcal{D}_\mu \mathcal{H}_\nu - \mathcal{D}_\nu \mathcal{H}_\mu - \frac{1}{\hbar c}[\mathcal{H}_\mu, \mathcal{H}_\nu] = \hbar c \cdot \mathcal{F}_{\mu\nu} \tag{3.33}$$

Thus, the complete Hamiltonian dynamics consists now of the previous conservation equation (3.24) and the present integrability condition (3.33). Both these equations are needed to deduce the energy-momentum law (3.7), or (3.12), respectively, with the Lorentz force given by its desired form (3.26), or (3.29), respectively. However, the conservation equation (3.24) additionally has a nice property, namely it ensures the validity of the Klein–Gordon equation for the wave function ψ,

$$\mathcal{D}^\mu \mathcal{D}_\mu \psi + \left(\frac{\mathcal{M}c}{\hbar}\right)^2 \cdot \psi = 0 \tag{3.34}$$

This is easily deduced from the original RSE (3.3) by differentiating once more and applying the conservation equation (3.24). Through this procedure it becomes clear that the KGE (3.34) inherits its conservative meaning from the RST, namely as a selection principle for those field configurations which obey the energy-momentum law. Perhaps this is the deeper meaning of all the wave equations: namely to equip the waves with an energy-momentum content $T_{\mu\nu}$ in such a way that the energy-momentum law (3.7) holds.

For dealing with concrete problems, it is sometimes convenient to split up the Hamiltonian $\mathcal{H}_\mu$ into its anti-Hermitian part $\mathcal{K}_\mu$ ($= -\overline{\mathcal{K}}_\mu$), the *kinetic field*, and its Hermitian part $\mathcal{L}_\mu$ ($= \overline{\mathcal{L}}_\mu$), the *localization field*:

$$\mathcal{H}_\mu = \hbar c(\mathcal{K}_\mu + \mathcal{L}_\mu) \tag{3.35}$$

The dynamical equations for $\mathcal{H}_\mu$, namely (3.24) and (3.33), can then be transcribed to the corresponding equations for the (anti-)Hermitian parts $\mathcal{K}_\mu$ and $\mathcal{L}_\mu$. In this way, the conservation equation reads for $\mathcal{K}_\mu$

$$\mathcal{D}^\mu \mathcal{K}_\mu + \{\mathcal{L}^\mu, \mathcal{K}_\mu\} = 0 \tag{3.36}$$

and for $\mathcal{L}_\mu$

$$\mathcal{D}^\mu \mathcal{L}_\mu + \mathcal{K}^\mu \mathcal{K}_\mu + \mathcal{L}^\mu \mathcal{L}_\mu = -\left(\frac{\mathcal{M}c}{\hbar}\right)^2 \tag{3.37}$$

Similarly, the integrability condition (3.33) reads for the anti-Hermitian part $\mathcal{K}_\mu$

$$\mathcal{D}_\mu \mathcal{K}_\nu - \mathcal{D}_\nu \mathcal{K}_\mu - [\mathcal{K}_\mu, \mathcal{K}_\nu] - [\mathcal{L}_\mu, \mathcal{L}_\nu] = \mathcal{F}_{\mu\nu} \tag{3.38}$$

and for the Hermitian part $\mathcal{L}_\mu$

$$\mathcal{D}_\mu \mathcal{L}_\nu - \mathcal{D}_\nu \mathcal{L}_\mu - [\mathcal{L}_\mu, \mathcal{K}_\nu] - [\mathcal{K}_\mu, \mathcal{L}_\nu] = 0 \tag{3.39}$$

4. COSMOLOGICAL PRINCIPLE

Schrödinger's treatment [1] of the Zitterbewegung apparently traces its origin to the intrinsic spin structure of the Dirac equation. However, if it turns out that the Zitterbewegung is a more general phenomenon, one can surely learn something about this curious effect by considering a concrete physical example *with neglect of spin*. For this purpose a simple model is obtained by considering a Friedmann–Robertson–Walker universe with the energy-momentum density $T_{\mu\nu}$, (3.10b), being carried by a *spinless* wave field $\psi(x)$, (3.3), respectively, by the intensity matrix $\mathcal{I}(x)$, (3.9). Since the energy-momentum density $T_{\mu\nu}$ of matter must contain the effects of the Zitterbewegung, but also influences the background geometry according to Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi \frac{L_p^2}{\hbar c} T_{\mu\nu} \tag{4.1}$$

(L_p is the Planck length), one can study the interplay between the scale parameter $\mathcal{R}$ of the Robertson–Walker line element [23]

$$ds^2 = d\Theta^2 - \mathcal{R}^2 \, dl^2 \tag{4.2}$$

and the Zitterbewegung. In other words, we expect that the "radius of the universe" $\mathcal{R} = \mathcal{R}(\Theta)$ as a function of cosmic time Θ will inherit some trembling component from the Zitterbewegung of matter, and a study of this trembling expansion of the universe will yield some insight into the Zitter effect.

Fortunately, the RW symmetry of such a universe simplifies the computations considerably. According to the cosmological principle [23], the matter distribution is taken to be homogeneous and isotropic, which forces the energy-momentum density $T_{\mu\nu}$ into the following simple form:

$$T_{\mu\nu} = U b_\mu b_\nu - P B_{\mu\nu} \tag{4.3}$$

Here the energy density $U = U(\Theta)$ and the pressure $P = P(\Theta)$ are functions of the cosmic time Θ, but they do not vary over the time slices $\Theta = \text{const}$ [whose 3-dimensional line element has been denoted by dl in Eq. (4.2)]. The Hubble flow vector has been denoted by b_μ

$$b_\mu \doteqdot \partial_\mu \Theta \tag{4.4}$$

$$(b^\mu b_\mu = +1)$$

and its orthogonal projector by $B_{\mu\nu}$ ($= B_{\nu\mu}$), i.e.,

$$B_{\mu\nu} b^\mu = 0 \tag{4.5a}$$

$$B_{\mu\nu} B^\nu_\lambda = B_{\mu\lambda} \tag{4.5b}$$

$$B^\mu_\mu = 3 \tag{4.5c}$$

Consequently the pseudo-Riemannian metric $g_{\mu\nu}$ of the underlying RW space-time is split up according to

$$g_{\mu\nu} = b_\mu b_\nu + B_{\mu\nu} \tag{4.6}$$

and a similar result also for the Einstein tensor on the left of Eq. (4.1):

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 3\left(H^2 - \frac{\sigma}{\mathcal{R}^2}\right) b_\mu b_\nu - \left(\frac{\sigma}{\mathcal{R}^2} - 3H^2 - 2\dot{H}\right) B_{\mu\nu} \quad (4.7)$$

As is well known, the RW symmetry can be realized by three different 4-geometries, classified by the topological index σ: closed universe ($\sigma = -1$), flat universe ($\sigma = 0$), and open universe ($\sigma = +1$).

With these arrangements, the Einstein equations (4.1) are simplified to the following two equations for the scale parameter $\mathcal{R}$ of the line element (4.2):

$$\ddot{\mathcal{R}} = -4\pi \frac{L_p^2}{\hbar c} \mathcal{R}\left(P + \frac{1}{3} U\right) \quad (4.8a)$$

$$H^2 \equiv \left(\frac{\dot{\mathcal{R}}}{\mathcal{R}}\right)^2 = \frac{\sigma}{\mathcal{R}^2} + \frac{8\pi}{3} \frac{L_p^2}{\hbar c} \cdot U \quad (4.8b)$$

The proper equation of motion for $\mathcal{R}$ is the first one, (4.8a), whereas the second one [the *Friedmann equation*, (4.8b)] plays the part of a first integral and therefore can serve as a constraint for the initial conditions. It should be obvious that the special physics of the model enters the Einstein equations (4.8) via the energy density U and pressure P and therefore we now have to specify these quantities in terms of the concepts of RST.

In order to obey the cosmological principle in the form (4.3), we adopt the Hamiltonian $\mathcal{H}_\mu$ to be of the corresponding symmetric form

$$\mathcal{H}_\mu = \mathcal{H} \cdot b_\mu \quad (4.9)$$

Indeed this assumption recasts the energy-momentum operator $\mathcal{T}_{\mu\nu}$, (3.21), into the required RW-symmetric form

$$\mathcal{T}_{\mu\nu} = \mathcal{U} \cdot b_\mu b_\nu - \mathcal{W} \cdot B_{\mu\nu} \quad (4.10)$$

with the energy operator $\mathcal{U}$ and pressure operator $\mathcal{W}$ as

$$\mathcal{U} = \frac{1}{2Mc^2} (\overline{\mathcal{H}} \cdot \mathcal{H} + (\mathcal{M}c^2)^2) \quad (4.11a)$$

$$\mathcal{W} = \frac{1}{2Mc^2} (\overline{\mathcal{H}} \cdot \mathcal{H} - (\mathcal{M}c^2)^2) \quad (4.11b)$$

Consequently, the energy density U and pressure P to be inserted into the Einstein equations (4.8) can be computed from this result by means of the usual trace recipe [cf. (3.10b)]:

$$U = \mathrm{tr}(\mathcal{I} \cdot \mathcal{U}) \quad (4.12a)$$

$$P = \mathrm{tr}(\mathcal{I} \cdot \mathcal{W}) \quad (4.12b)$$

From this it should become clear that different realizations of RST will lead to different results for U, (4.12a), and P, (4.12b), and therefore will imply a different expansion behavior $\mathcal{R}(\Theta)$ of the universe according to the Einstein equations (4.8). In particular, concerning the interplay of the Zitterbewegung of matter and the collapse of the universe, the Einsteinian expansion dynamics reacts very sensitively and can therefore be taken as a probe to feel the trembling effect. In this respect, we shall now study two different realizations of RST.

5. COMPLEX ONE-DIMENSIONAL REALIZATION

The simplest situation is encountered when the wave function ψ is an ordinary complex number, i.e., when $\psi(x)$ is a section of a complex line bundle (with typical fiber $\mathbb{C}^1$). This case is equivalent to the ordinary Klein–Gordon theory and has already been studied in two previous papers [8, 15]. It is therefore sufficient here to present the results only insofar as they are relevant for the question of Zitterbewegung. In fact we shall readily show that the Zitterbewegung of the ordinary Klein–Gordon theory actually is not sufficient to prevent the universe from collapsing (as was the case for the Dirac theory [20]).

5.1. $\mathbb{C}^1$-Realization of RST

Since only one complex dimension is available, the Hamiltonian is simplified to an ordinary $\mathbb{C}^1$-valued 1-form and may therefore be split up into its real and imaginary parts as [cf. (3.35)]

$$\mathcal{H}_\mu = \hbar c(-iK_\mu + L_\mu) \tag{5.1}$$

Here the kinetic field K_μ and localization field L_μ are ordinary (i.e., $\mathbb{R}^1$-valued) 1-forms. Since the complex numbers do commute, all the commutators occuring in the equations of motion must necessarily vanish and, e.g., the curl relation (3.39) for the localization field $\mathcal{L}_\mu$ simply reads now

$$\nabla_\mu L_\nu - \nabla_\nu L_\mu = 0 \tag{5.2}$$

However, this says that L_μ must be a gradient field, which therefore gives rise to the introduction of some "*amplitude field*" $L(x)$:

$$L_\mu = \frac{\partial_\mu L}{L} \tag{5.3}$$

The significance of this amplitude field is readily eludicated by the observation that the conservation equation for the kinetic field $\mathcal{K}_\mu$, (3.36), can be recast into the following form:

$$\nabla^{\mu}(L^2 \cdot K_{\mu}) \equiv 0 \tag{5.4}$$

Clearly this is nothing else than the 1-dimensional realization of the charge conservation law (3.4), where of course the gauge-covariant derivative D_{μ} must be replaced by the coordinate-covariant derivative ∇_{μ} because of the Abelian character of the gauge group $U(1)$ to be applied here. Thus the single ($a = 1$) current density j_{μ}, (3.10a), is immediately read off from Eq. (5.4) as

$$j_{\mu} \sim L^2 \cdot K_{\mu} \tag{5.5}$$

(up to a constant prefactor). This result is also consistent with the definition of the single ($a = 1$) velocity operator v_{μ}, (3.20),

$$v_{\mu} = -\frac{1}{2Mc^2}(\tau \cdot \mathcal{H}_{\mu} - \overline{\mathcal{H}}_{\mu} \cdot \tau) \tag{5.6}$$

which yields

$$v_{\mu} = \frac{\hbar}{Mc} K_{\mu} \tag{5.7}$$

due to the fact that the single generator ($a = 1$) of the 1-dimensional gauge group $U(1)$ is $\tau = -i$. Furthermore, we conclude from the relativistic von Neumann equation for the (1-dimensional) intensity matrix $\mathcal{I}$,(3.9), that $\mathcal{I}$ can be identified with the square of the amplitude field, i.e.,

$$\mathcal{I} = L^2 \tag{5.8}$$

Thus the general definition for the current density j_{μ}, (3.10a), yields

$$j_{\mu} = \frac{\hbar}{Mc} L^2 \cdot K_{\mu} \tag{5.9}$$

in agreement with the preliminary guess (5.5).

5.2. Conventional Klein–Gordon Theory

The present $\mathbb{C}^1$-realization of RST is equivalent to the Klein–Gordon theory. In ordinary Klein–Gordon theory, the current density j_{μ} is expressed by means of the wave function ψ and its derivative $\mathcal{D}_{\mu}\psi$. Therefore, in order to demonstrate the equivalence with the present form (5.9), we first have to construct the $\mathbb{C}^1$-valued wave function $\psi(x)$. This may be done simply by a formal integration of the RSE (3.3), which reads for the present 1-dimensional case

$$(\partial_{\mu} - iA_{\mu})\psi = (-iK_{\mu} + L_{\mu})\psi \tag{5.10}$$

Obviously, the formal solution here can be written in the form

$$\psi(x) = L(x) \cdot \exp\left(-i \int^{x} (K_\mu - A_\mu)\, dx\right) \tag{5.11}$$

But for the sake of consistency we have to convince ourselves that the integral actually is well defined for any endpoint x of space-time. This requirement can be satisfied only if the vector integrand $(K_\mu - A_\mu)$ is a gradient field ($\partial_\mu \alpha$, say):

$$K_\mu - A_\mu = \partial_\mu \alpha \tag{5.12}$$

For this case, the wave function $\psi(x)$ then adopts the well-known form

$$\psi(x) = L(x) \cdot e^{-i\alpha(x)} \tag{5.13}$$

and obeys the Klein–Gordon equation (3.34),

$$\mathscr{D}^\mu \mathscr{D}_\mu \psi + \left(\frac{Mc}{\hbar}\right)^2 \psi = 0 \tag{5.14}$$

$$(\mathscr{D}_\mu \psi \doteqdot \partial_\mu \psi - iA_\mu \psi)$$

However, the crucial gradient condition (5.12) does hold because the integrability condition for the kinetic field (3.38) reads for the 1-dimensional case

$$\nabla_\mu K_\nu - \nabla_\nu K_\mu = F_{\mu\nu} \tag{5.15}$$

Here we have used the decomposition (3.27) for the present 1-dimensional situation as

$$\mathscr{F}_{\mu\nu} = F_{\mu\nu} \cdot \tau \Rightarrow -iF_{\mu\nu} \tag{5.16}$$

(and similarly for $\mathscr{A}_\mu$). Clearly, the curl relation (5.15) does not yet allow us to consider the kinetic field K_μ as a gradient field, but the field strength $F_{\mu\nu}$ has already been defined as the curl of a vector potential A_μ [cf. (3.8)]:

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \tag{5.17}$$

and therefore the combination of Eqs. (5.15) and (5.17) immediately leads us to the desired gradient condition (5.12). *Thus the complex 1-dimensional realization of RST is actually equivalent to the ordinary Klein–Gordon theory!* As a consequence, all quantities of the conventional Klein–Gordon formalism (usually expressed in terms of the wave function ψ and vector potential A_μ) can be translated into the formalism of RST and thus can be expressed in terms of the kinetic and localization fields. For instance, consider the current density j_μ, (5.9). According to the formal *isomorphism* between (the present realization of) RST and the conventional Klein–Gordon theory, it must be possible to reexpress j_μ in terms of the wave function ψ and its covariant derivative $\mathcal{D}_\mu\psi$ [specified in (5.14)]. Indeed, the desired alternative form for the current j_μ is easily found as

$$j_\mu = \frac{i\hbar}{2Mc} (\overset{*}{\psi} \cdot \mathcal{D}_\mu\psi - \psi \cdot \mathcal{D}_\mu\overset{*}{\psi}) \tag{5.18}$$

But this can immediately be transcribed to the RST form (5.9) by means of the RSE (3.3) and the decomposition of the Hamiltonian $\mathcal{H}_\mu$, (5.1). Of course the present result (5.18) is the conventional form for j_μ to be found in any textbook dealing with the Klein–Gordon theory [e.g., 2, 3]. Observe also that the physical densities (as the proper observable quantities in RST) must be gauge-invariant. The gauge invariance of the RST current j_μ, (5.9), follows immediately from the gauge invariance of the (1-dimensional) Hamiltonian $\mathcal{H}_\mu$, (5.1), with respect to the (1-dimensional) $U(1)$ transformations $\mathcal{S}$,

$$\mathcal{H}'_\mu = \mathcal{S} \cdot \mathcal{H}_\mu \cdot \mathcal{S}^{-1} \equiv \mathcal{H}_\mu \tag{5.19}$$

$$(\mathcal{S}(x) = \exp(-ia(x)) \in U(1))$$

Equivalently, the gauge invariance of the Klein–Gordon form for j_μ, (5.18), follows from the fact that both the wave function $\psi(x)$ and its covariant derivative $\mathcal{D}_\mu\psi$ transform homogeneously:

$$\psi' \doteqdot \mathcal{S} \cdot \psi \equiv e^{-ia} \cdot \psi = L \cdot e^{-i(\alpha+a)} \doteqdot L \cdot e^{-i\alpha'} \tag{5.20a}$$

$$\mathcal{D}'_\mu\psi' = e^{-ia} \cdot \mathcal{D}_\mu\psi \tag{5.20b}$$

This implies that the phase $\alpha(x)$ of the general wave function ψ, (5.13), is changed additively by the gauge parameter a, (5.19), i.e.,

$$\alpha(x) \Rightarrow \alpha'(x) = \alpha(x) + a(x) \tag{5.21}$$

It is well known that it is just this *inhomogeneous* transformation rule (5.21) which combines with the corresponding *inhomogeneous* law for the connection $\mathcal{A}_\mu$,

$$\mathcal{A}'_\mu = \mathcal{S} \cdot \mathcal{A}_\mu \cdot \mathcal{S}^{-1} + \mathcal{S} \cdot \partial_\mu\mathcal{S}^{-1} \tag{5.22}$$

i.e., for the present case

$$A'_\mu = A_\mu - \partial_\mu a \tag{5.23}$$

in order to produce the *homogeneous* rule for the covariant derivative $\mathcal{D}_\mu\psi$, (5.20b).

5.3. Zitterbewegung

With all these results in mind, we are now well prepared to clarify the question of Zitterbewegung. If this effect should really exist in the Klein–

Gordon theory, one would expect that there is some variable of oscillatory character which carries the information about the Zitterbewegung. Clearly the phase factor $e^{-i\alpha}$ suggests itself as such a "trembling variable." However, as the preceding discussion of the current density $j_\mu(x)$ demonstrates, the phase $\alpha(x)$ does not enter the physical densities and therefore we now have to compute the energy density U, (4.12a), and pressure P, (4.12b), of matter in order to insert this into the Einstein equations (4.8) for probing the reaction of the scale parameter $\mathcal{R}$ with respect to the presence or absence of Zitter components.

According to the cosmological principle, the general Hamiltonian $\mathcal{H}_\mu$ of the ordinary Klein–Gordon theory, (5.1), must reduce to the homogeneous and isotropic form (4.9) with the scalar $\mathcal{H}$ being given by

$$\mathcal{H} = \hbar c(-iK + \Lambda) \tag{5.24}$$

provided we put for the kinetic field $\mathcal{K}_\mu$ and localization field $\mathcal{L}_\mu$

$$K_\mu = K \cdot b_\mu \tag{5.25a}$$

$$L_\mu = \Lambda \cdot b_\mu \tag{5.25b}$$

$$[\Lambda \equiv \dot{L}/L; \text{ cf. } (5.3)]$$

Furthermore, in view of the fact that for the 1-dimensional case the intensity matrix $\mathcal{I}(x)$ coincides with the square of the amplitude field $\mathrm{L}(x)$ [cf. (5.8)], the energy density U, (4.12a), becomes

$$U \Rightarrow U_{\mathrm{KG}} = \frac{\hbar^2}{2M}\left[K^2 + \Lambda^2 + \left(\frac{Mc}{\hbar}\right)^2\right] \cdot L^2 \tag{5.26}$$

and similarly we get for the pressure P, (4.12b),

$$P \Rightarrow P_{\mathrm{KG}} = \frac{\hbar^2}{2M}\left(K^2 + \Lambda^2 - \left(\frac{Mc}{\hbar}\right)^2\right) \cdot L^2 \tag{5.27}$$

This can now be substituted into the Einstein equations (4.8), but before we can integrate this Einsteinian system, we have to supplement it by the corresponding equations of motion for the kinetic field K and localization field Λ. Here, the integrability conditions (3.38) and (3.39) for $\mathcal{K}_\mu$ and $\mathcal{L}_\mu$ are trivially satisfied (observe $\mathcal{F}_{\mu\nu} \equiv 0$) on account of the cosmological principle (5.25) plus the Abelian character of the 1-dimensional case. Thus, we are left with the conservation equations (5.4) for K_μ and (3.37) for L_μ.

First consider the charge conservation law (5.4) with the current density j_μ being given in its final form by Eq. (5.9). Due to the cosmological principle, this current density must be of the form

$$j_\mu = I \cdot b_\mu \tag{5.28}$$

$$\left(I \doteqdot \frac{\hbar}{Mc} L^2 \cdot K\right)$$

Furthermore, the charge conservation

$$\nabla^\mu j_\mu \equiv 0 \tag{5.29}$$

as the 1-dimensional specialization of the general case (3.4), says that for the scalar factor I, (5.28),

$$\dot{I} + 3HI = 0 \tag{5.30}$$

But remembering the definition of the Hubble expansion rate H in terms of the scale parameter $\mathcal{R}$, (4.8b), we can immediately integrate this to yield

$$I \equiv \frac{\hbar}{Mc} \cdot L^2 \cdot K = \frac{\hbar}{Mc} \frac{I_*}{\mathcal{R}^3} \tag{5.31}$$

$$(I_* = \text{const})$$

Finally, consider the general equation of motion for the localization field $\mathcal{L}_\mu$, (3.37). Since, for the ordinary Klein–Gordon theory (5.1), the localization field turned out to be a gradient field (5.3), the equation of motion for $\mathcal{L}_\mu$, (3.37), effectively is a wave equation for the amplitude field $L(x)$:

$$\Box L + \left(\left(\frac{Mc}{\hbar}\right)^2 - K^\mu K_\mu\right) \cdot L = 0 \tag{5.32}$$

$$(\Box \doteqdot \nabla^\mu \nabla_\mu)$$

For the present homogeneous and isotropic case, where the amplitude field $L(x)$ exclusively depends upon the cosmic time Θ [$\rightsquigarrow L = L(\Theta)$], this wave equation simplifies to

$$\ddot{L} + 3H\dot{L} + \left[\left(\frac{Mc}{\hbar}\right)^2 - K^2\right] \cdot L = 0 \tag{5.33}$$

Thus the Einstein–Klein–Gordon system is complete and its solutions exhibit the following features.

In a closed RW universe ($\sigma = -1$), there exist solutions for the Einsteinian system (4.8), but the corresponding radius $\mathcal{R}$, energy density U_{KG}, (5.26), and pressure P_{KG}, (5.27), are very insensitive with respect to the

Zitterbewegung (Figs. 1 and 2). The initial conditions are chosen in such a way that initially ($\mathcal{R} \gtrsim 0$) one has the equation of state $P \approx -U \approx -\frac{1}{2}Mc^2L^2$, which implies inflationary expansion [$\ddot{\mathcal{R}} > 0$; see Eq. (4.8a)]. But this equation of state cannot persist beyond the initial phase and the universe must ultimately contract to a point in a highly singular behavior ($P \approx U \Rightarrow +\infty$).

Though being present very weakly in the external quantities U and P, there occurs nevertheless a violent Zitterbewegung for the amplitude field L (Fig. 3) and kinetic field K (Fig. 4). The origin of this kind of trembling behavior is best be seen by truncating the amplitude equation (5.33) to

$$\ddot{L} + \left(\frac{Mc}{\hbar}\right)^2 \cdot L = 0 \tag{5.34}$$

which would admit harmonic oscillations of frequency $\sqrt{Mc/\hbar}$. However, such oscillations would imply also the existence of zeros for L, which is forbidden by the conservation law (5.31). As a compromise, the amplitude field adopts minimal values (in place of the forbidden zeros; cf. Fig. 3), and

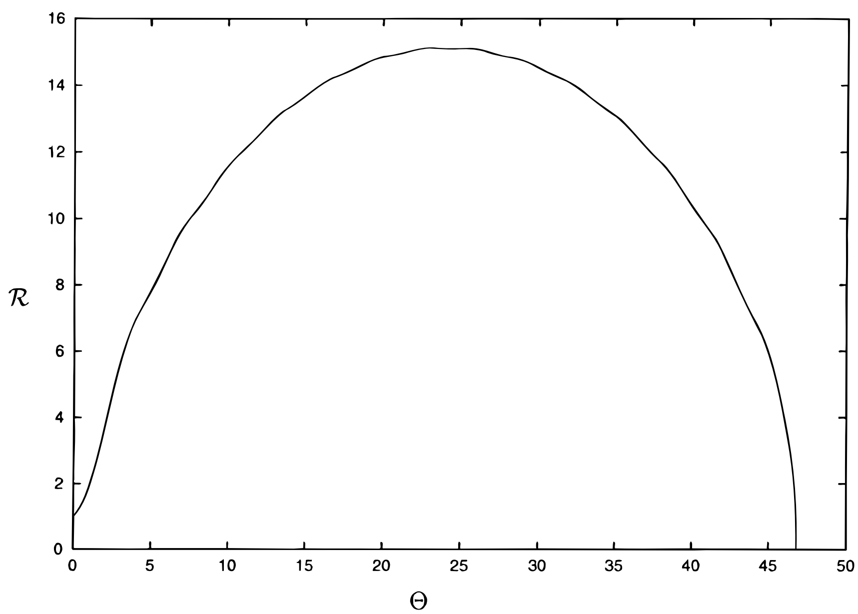

Fig. 1. Inflation and collapse of the closed universe ($\sigma = -1$). The initial inflation phase ($\ddot{\mathcal{R}} > 0$) is due to a special choice of the initial conditions leading to the equation of state $P \approx -U$, which, however, is dynamically converted to $P \approx +U$ for the collapse phase ($\mathcal{R} \to 0$, $\ddot{\mathcal{R}} \to -\infty$). The Zitterbewegung causes only small oscillations of the universe's radius $\mathcal{R}$. All lengths ($\mathcal{R}$, Θ, . . .) are measured in units of the Compton length $\hbar/Mc$).

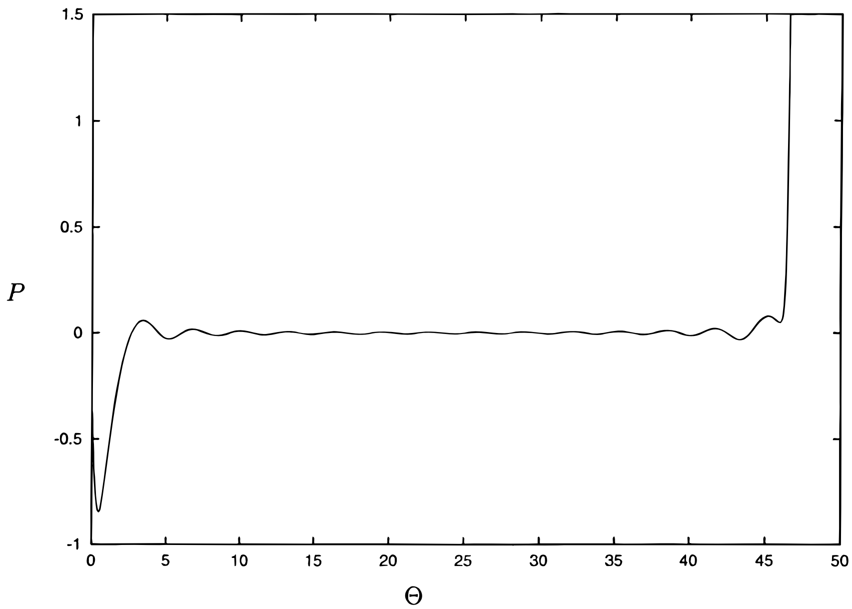

Fig. 2. Equation of state. Apart from small oscillations due to the Zitterbewegung, the pressure P becomes zero for a large part of the universe's lifetime. According to the work-energy theorem $d(U \cdot \mathcal{R}^3) = -P\, d(\mathcal{R}^3)$, being implied by the Einsteinian system (4.8), the energy density varies as $U \sim \mathcal{R}^{-3}$ for the intermediate phase with vanishing pressure ($P \sim 0$). The pressure P and energy density U become infinite at the collapse point.

in order to simultaneously obey the conservation law (5.31) the kinetic field K must adopt pulsating behavior at these "almost-zeros" of L (cf. Fig. 4).

6. REAL, TWO-DIMENSIONAL REALIZATION

Any complex number, such as, e.g., the wave function $\psi(x)$ of the conventional Klein–Gordon theory (5.11), can be considered as a pair of real numbers, φ_1 and φ_2, say:

$$\psi(x) = \varphi_1(x) + i\varphi_2\,(x) \tag{6.1}$$

Thus it may seem that one cannot gain anything new when we take now as the typical fiber for our vector bundle (of wave functions) the 2-dimensional real space $\mathbb{R}^2$ in place of the complex 1-dimensional space $\mathbb{C}^1$ which is applied for the conventional Klein–Gordon theory. However, such a supposition misses the point for the following reason: in RST the relevant objects are not the *wave functions* $\psi(x)$ (as the sections of the appropriate vector bundles), but the relevant objects are here the operators (operator-valued sections) acting over the corresponding vector fibers. Whereas an operator

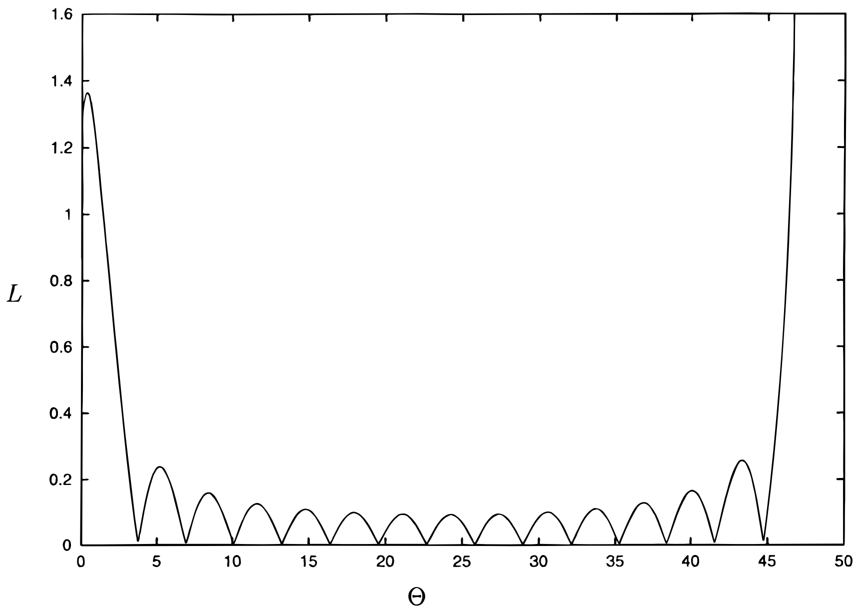

Fig. 3. Zitterbewegung of amplitude field. The amplitude field $L(\Theta)$ cannot perform full oscillations (including negative values) because the conservation law (5.31) forbids the occurrence of zeros for L.

acting over $\mathbb{C}^1$ is also a complex number and therefore has only *two* field degrees of freedom [cf. the Hamiltonian $\mathcal{H}_\mu$, (5.1)], the operators acting over the fiber space $\mathbb{R}^2$ have in general *two* $\times$ *two* $=$ *four* independent matrix elements and therefore the present $\mathbb{R}^2$-realization of RST must be expected to be equipped with a much richer structure than the ordinary Klein–Gordon theory! In particular, one can describe here the properties of matter by means of an intensity matrix $\mathcal{I}$, in place of the poorer case of a wave function ψ as in the Klein–Gordon theory. As we shall readily see, this additional structure makes the Zitterbewegung more intricate.

6.1. Physical Densities

In contrast to the poor case of a (1×1) matrix for $\mathcal{I}$ in Klein–Gordon theory (5.8), the (symmetric) intensity matrix $\mathcal{I}$ $(= \overline{\mathcal{I}})$ is now in general a linear combination of three symmetric basis operators. For a convenient choice of this operator basis we may take two orthogonal projectors $\mathcal{P}_+$ $(= \mathcal{P}_+^2 = \overline{\mathcal{P}}_+)$ and $\mathcal{P}_-$ $(= \mathcal{P}_-^2 = \overline{\mathcal{P}}_-)$:

$$\mathcal{P}_+ \cdot \mathcal{P}_- = 0 \tag{6.2a}$$

$$\mathcal{P}_+ + \mathcal{P}_- = \mathbf{1} \tag{6.2b}$$

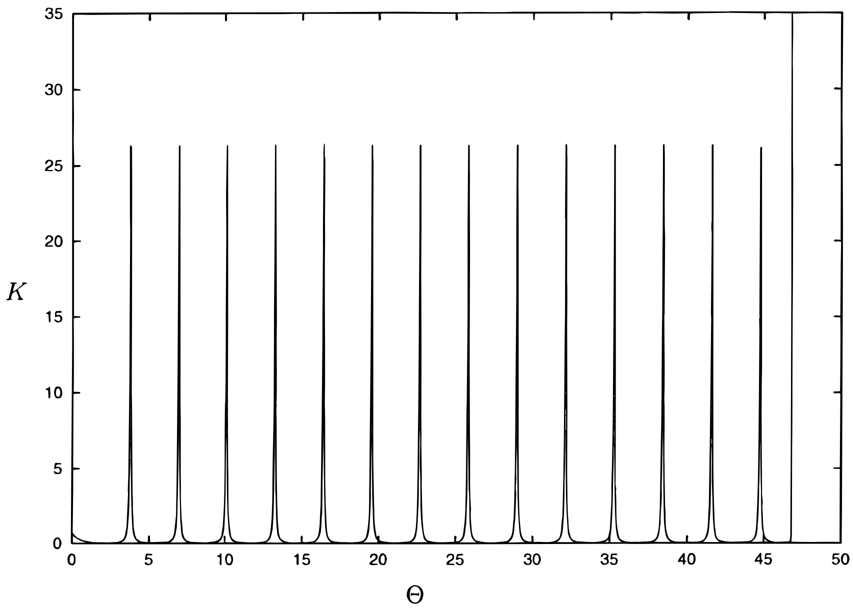

Fig. 4. Pulsations of the kinetic field. The conservation law (5.31) forces the kinetic field to perform very short pulses at the "almost zeros" of the amplitude field L. Observe the regularity of the pulses over a wide range of the universe's size $\mathcal{R}$. The kinetic field K becomes infinite at the collapse point.

These projectors tentatively might be associated to the particle and antiparticle states. The third symmetric operator is the permutator Π. All symmetric matrices can now be expanded with respect to this 3-dimensional basis $\{\mathcal{P}_+, \mathcal{P}_-, \Pi\}$, e.g., the intensity matrix $\mathcal{I}$:

$$\mathcal{I} = \rho_+ \cdot \mathcal{P}_+ + \rho_- \cdot \mathcal{P}_- + s \cdot \Pi \tag{6.3}$$

A fourth independent operator exists (τ, say) which is antisymmetric ($\bar{\tau} = -\tau$) and therefore can be taken as the generator for the gauge group $SO(2)$, which is isomorphic to the Klein–Gordon counterpart $U(1)$ of the preceding section. Thus the gauge potential $\mathcal{A}_\mu$, (3.17), is simplified to

$$\mathcal{A}_\mu = A_\mu \cdot \tau \tag{6.4}$$

and analogously for its curvature $\mathcal{F}_{\mu\nu}$, (3.27),

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} \cdot \tau \tag{6.5}$$

As a special representation of the complete basis $\{\mathcal{P}_+, \mathcal{P}_-, \Pi, \tau\}$ we could take the following:

$$\mathcal{P}_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \qquad \mathcal{P}_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.6}$$

$$\Pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The densities $\{\rho_+, \rho_-, s\}$ can then be recovered from the intensity matrix $\mathcal{I}$, (6.3), in an obvious way,

$$\rho_+ = \mathrm{tr}(\mathcal{I} \cdot \mathcal{P}_+) \tag{6.7a}$$

$$\rho_- = \mathrm{tr}(\mathcal{I} \cdot \mathcal{P}_-) \tag{6.7b}$$

$$2s = \mathrm{tr}(\mathcal{I} \cdot \Pi) \tag{6.7c}$$

As intuitive as the picture of this interplay between *particle and antiparticle* states may be in relativistic quantum theory, it seems more adequate not to think here in these categories, but rather in terms of the concepts of "*internal*" and "*external*" motions of the relativistic particle. The reason for this is immediately seen by inspection of the gauge transformations upon the intensity matrix $\mathcal{I}$:

$$\mathcal{I}' = \mathcal{S} \cdot \mathcal{I} \cdot \mathcal{S}^{-1} \tag{6.8}$$

$$(\mathcal{S} = \exp(a \cdot \tau) \in SO(2))$$

where the gauge element $\mathcal{S} \in SO(2)$ is of course the counterpart of the corresponding $U(1)$ element in the Klein–Gordon theory [cf. (5.19)]. The change of gauge (6.8) transforms the densities $\{\rho_+, \rho_-, s\}$ as follows:

$$\rho'_+ = \rho_+ \cdot \cos^2 a + \rho_- \cdot \sin^2 a + s \cdot \sin 2a \tag{6.9a}$$

$$\rho'_- = \rho_- \cdot \cos^2 a + \rho_+ \cdot \sin^2 a - s \cdot \sin 2a \tag{6.9b}$$

$$s' = -\frac{1}{2}(\rho_+ - \rho_-) \sin 2a + s \cdot \cos 2a \tag{6.9c}$$

As expected, the particle density ρ_+ is mixed up here with the antiparticle density ρ_- in such a way that their sum $\rho = \rho_+ + \rho_-$ as the trace of the intensity matrix $\mathcal{I}$, (6.3), remains invariant, i.e.,

$$\rho' \equiv \rho = \mathrm{tr}\mathcal{I} \tag{6.10}$$

This result compares now to the invariance of the intensity matrix $\mathcal{I}$, (5.8), in the preceding Klein–Gordon theory. Therefore, if we want to compare the preceding $\mathbb{C}^1$-realization with the present $\mathbb{R}^2$-realization of RST, we should collect the gauge-invariant densities into one subset (the "external" densities) and the gauge dependent densities into another subset (the "internal" densities) so that the external objects become the direct analogues of the Klein–Gordon objects and the internal objects become responsible for the differences of the theories. But observe here that these differences refer to the fact that the $\mathbb{R}^2$-realization is capable of dealing also with particle–antiparticle *mixtures*, whereas the $\mathbb{C}^1$-realization can deal only with the *superposition* of particle and antiparticle states. Thus we see that the $\mathbb{C}^1$-realization is embedded as a special case into the more general $\mathbb{R}^2$-realization, namely in the form of the subset of pure states. Consequently, neglect of the internal degrees of freedom of the present $\mathbb{R}^2$-realization makes it identical to the preceding $\mathbb{C}^1$-realization (i.e. Klein–Gordon theory). Superfluous to say that the internal motion will further complicate the notorious Zitterbewegung.

According to this philosophy, we introduce the sum of densities ρ, (6.10), and their difference q,

$$q \doteq \rho_+ - \rho_- \tag{6.11}$$

in place of ρ_+ and ρ_- and then find from (6.9) that the internal variables form an $SO(2)$ doublet:

$$q' = q \cdot \cos 2a + 2s \cdot \sin 2a \tag{6.12a}$$

$$2s' = 2s \cdot \cos 2a - q \cdot \sin 2a \tag{6.12b}$$

Correspondingly, an operator doublet $\{Q, \Pi\}$ is obtained by putting

$$Q \doteq \mathcal{P}_+ - \mathcal{P}_- \tag{6.13}$$

which then yields the following transformation formulas:

$$Q' \doteq \mathcal{S} \cdot Q \cdot \mathcal{S}^{-1} = Q \cdot \cos 2a - \Pi \cdot \sin 2a \tag{6.14a}$$

$$\Pi' \doteq \mathcal{S} \cdot \Pi \cdot \mathcal{S}^{-1} = \Pi \cdot \cos 2a + Q \cdot \sin 2a \tag{6.14b}$$

Consequently, the intensity matrix $\mathcal{I}$ can now be decomposed into its gauge-invariant and covariant constituents:

$$\mathcal{I} = \frac{1}{2}\rho \cdot \mathbf{1} + \frac{1}{2} q \cdot Q + s \cdot \Pi \tag{6.15}$$

Of course, the gauge covariance must also apply to the corresponding covariant derivatives, i.e., we must have, in analogy to the transformation behavior (6.12),

$$D'_\mu q' = D_\mu q \cdot \cos 2a + D_\mu(2s) \cdot \sin 2a \tag{6.16a}$$

$$D'_\mu(2s') = D_\mu(2s) \cdot \cos 2a - D_\mu q \cdot \sin 2a \tag{6.16b}$$

This is easily verified by means of the $SO(2)$ covariant derivatives for the densities, to be defined through

$$D_\mu q \doteqdot \partial_\mu q + 4A_\mu \cdot s \tag{6.17a}$$

$$D_\mu s \doteqdot \partial_\mu s - A_\mu \cdot q \tag{6.17b}$$

Observe here also that the change of the gauge potential A_μ, (5.23), is the same for both realizations on account of the isomorphism of both gauge groups $U(1)$ and $SO(2)$.

But an important difference between the two realizations lies in the fact that the $\mathbb{C}^1$-realization *must* always deal with a pure state $\psi(x)$, whereas the $\mathbb{R}^2$-realization can also deal with a mixture. Observe that a pure state can always be considered as a special case of a mixture, namely when the intensity matrix $\mathcal{I}$ degenerates into the tensor product of the pure state ψ (i.e., $\mathcal{I} \Rightarrow \psi \otimes \bar{\psi}$). The necessary and sufficient condition for this to occur is the following *Fierz identity* [10] for the intensity matrix $\mathcal{I}$:

$$\mathcal{I}^2 = \rho \cdot \mathcal{I} \tag{6.18}$$

This condition reads in terms of the densities

$$\rho^2 = q^2 + (2s)^2 \tag{6.19}$$

which is consistent with the gauge invariance [cf. (6.10) and (6.12)].

6.2. Hamiltonian Dynamics

In RST, the motion of matter is governed by the relativistic von Neumann equation (3.9), but before this equation can be converted to an equation of motion for the densities, one must know the Hamiltonian $\mathcal{H}_\mu$ as a solution of its field equations, namely the conservation equation (3.24) and the integrability condition (3.33). Therefore let us first decompose $\mathcal{H}_\mu$ with respect to the new operator basis $\{\mathbf{1}, Q, \Pi, \tau\}$ and then determine the equations of motion for the coefficients of that decomposition.

The kinetic field $\mathcal{K}_\mu$ as the antisymmetric part of the Hamiltonian $\mathcal{H}_\mu$ has a very simple structure because we have here only one antisymmetric basis operator (τ), similarly as in the Klein–Gordon theory (5.1):

$$\mathcal{K}_\mu = K_\mu \cdot \tau \tag{6.20}$$

But the localization field has a rather nontrivial decomposition:

$$\mathcal{L}_\mu = L_\mu \cdot \mathbf{1} + \frac{1}{2} l_\mu \cdot Q + N_\mu \cdot \Pi \tag{6.21}$$

which degenerates to the Klein–Gordon case [$\mathcal{L}_\mu = \mathcal{L}_\mu \cdot 1$; see (5.1)] only if the internal part is neglected! The general curl relations (3.39) again require here the existence of an amplitude field $L(x)$ just as in the Klein–Gordon theory [cf. (5.3)]:

$$L_\mu = \frac{\partial_\mu L}{L} \tag{6.22}$$

where this amplitude field refers now to the *external* motion. The remaining curl relations for the internal part of $\mathcal{L}_\mu$ are easily deduced from the general equation (3.39) as

$$D_\mu l_\nu - D_\nu l_\mu = 4(K_\mu N_\nu - K_\nu N_\mu) \tag{6.23a}$$

$$D_\mu N_\nu - D_\nu N_\mu = l_\mu K_\nu - l_\nu K_\mu \tag{6.23b}$$

Clearly, since both internal vector fields l_μ and N_μ are $SO(2)$ gauge objects,

$$l'_\mu = l_\mu \cdot \cos 2a + (2N_\mu) \cdot \sin 2a \tag{6.24a}$$

$$2N'_\mu = 2N_\mu \cdot \cos 2a - l_\mu \cdot \sin 2a \tag{6.24b}$$

(and similarly for the covariant derivatives), one must always apply their $SO(2)$ covariant derivatives (D), i.e.,

$$D_\mu l_\nu \doteqdot \nabla_\mu l_\nu + 4A_\mu \cdot N_\nu \tag{6.25a}$$

$$D_\mu N_\nu \doteqdot \nabla_\mu N_\nu - A_\mu \cdot l_\nu \tag{6.25b}$$

The analogous curl relation (3.38) for the kinetic field $\mathcal{K}_\mu$, (6.20), must of course be very simple again and is found to be

$$\nabla_\mu K_\nu - \nabla_\nu K_\mu = F_{\mu\nu} + l_\mu N_\nu - l_\nu N_\mu \tag{6.26}$$

This differs from its Klein–Gordon counterpart, (5.15), by the wedge product of the internal vectors l_μ and N_ν which is added to the electromagnetic field $F_{\mu\nu}$ on the right-hand side.

Next, consider the conservation equation for $\mathcal{K}_\mu$, (3.36). Since the anti-commutators of the Hermitian operators Q and Π and of the anti-Hermitian operator τ vanish, it is only the gradient part L_μ, (6.22), of the localization field $\mathcal{L}_\mu$, (6.21), which becomes active and thus Eq. (3.36) directly leads to the following conservation law:

$$\nabla^\mu\, {}^{(1)}j_\mu \equiv 0 \tag{6.27a}$$

$${}^{(1)}j_\mu = \frac{\hbar}{Mc} L^2 \cdot K_\mu \tag{6.27b}$$

This is exactly the same conserved current as was found for the Klein–Gordon theory; cf. (5.9). However, in addition to the conserved current (6.27), existing also in the $\mathbb{C}^1$-realization, we have here a second conserved current in the $\mathbb{R}^2$-realization, namely the one given by the original definition (3.10a). In order to see that this latter current is really different from the former ${}^{(1)}j_\mu$, (6.27b), we explicitly compute the velocity operator v_μ, (5.6), by means of the Hamiltonian $\mathscr{H}_\mu$ as given by Eqs. (6.20) plus (6.21) and then find from (3.10a)

$$^{(2)}j_\mu \doteqdot \mathrm{tr}(\mathscr{I} \cdot v_\mu) = \frac{\hbar}{Mc}(\rho \cdot K_\mu + s \cdot l_\mu - q \cdot N_\mu) \tag{6.28}$$

Evidently this current $^{(2)}j_\mu$ reduces to the previous current $^{(1)}j_\mu$, (6.27b), through neglect of the internal motion, where the density ρ can be identified with the square of the amplitude field L^2. Clearly it is suggestive to consider the present current $^{(2)}j_\mu$ as the counterpart of Dirac's current $^{(D)}j_\mu$, (2.14)–(2.15), which embraces both the convection part $^{(C)}j_\mu$, (2.20a), and the polarization part $^{(P)}j_\mu$, (2.20b). Correspondingly, one feels strongly tempted here to define the internal part $^{(i)}j_\mu$ of the total current $^{(2)}j_\mu$ through

$$^{(i)}j_\mu = {}^{(2)}j_\mu - {}^{(1)}j_\mu = \frac{\hbar}{Mc}((\rho - L^2) \cdot K_\mu + s \cdot l_\mu - q \cdot N_\mu) \tag{6.29}$$

Observe here that, though this current $^{(i)}j_\mu$ is due to the internal motion and therefore is composed of $SO(2)$ gauge objects, it is actually gauge invariant and obeys a separate conservation law

$$\nabla^\mu \, {}^{(i)}j_\mu \equiv 0 \tag{6.30}$$

[The gauge invariance of $^{(i)}j_\mu$ immediately is implied by the gauge behavior of the Hamiltonian coefficients l_μ, N_μ, (6.24), and of the internal densities s, q, (6.12).] The associated conserved "charge" z_* arises by integration over some 3-dimensional hypersurface (S):

$$z_* = \int_{(S)} {}^{(i)}j_\mu \cdot dS^\mu \tag{6.31}$$

and can be taken as a global measure for the strength of excitation of the internal motion (Klein–Gordon theory: $z_* = 0$). It is just because of this conservation law (6.31) that any mixture with $z_* \neq 0$ cannot evolve into a pure state (which has $z_* = 0$). However, those mixtures with $z_* = 0$ ($\rightsquigarrow$ "*quasi-pure states*") can well decay to a truly pure state; see below for the discussion of the quantum jumps.

Finally, in order to close the Hamiltonian dynamics, we have to consider also the conservation equation for the localization field $\mathscr{L}_\mu$, (3.37). Indeed, this is a very important part of the whole dynamics because it is the RST analogue of the energy eigenvalue equation of conventional quantum theory. In order to see this in more detail, we write down the general equation (3.37) in terms of the localization coefficients L_μ, l_μ, N_μ, (6.21), and find

$$\Box L + \left(\left(\frac{Mc}{\hbar} \right)^2 - K^\mu K_\mu + N^\mu N_\mu + \frac{1}{4} l^\mu l_\mu \right) \cdot L = 0 \qquad (6.32a)$$

$$D^\mu (L^2 \cdot N_\mu) = 0 \qquad (6.32b)$$

$$D^\mu (L^2 \cdot l_\mu) = 0 \qquad (6.32c)$$

The comparison of the present amplitude equation (6.32a) to its Klein–Gordon analogue (5.32) again demonstrates the presence of the internal motion in the $\mathbb{R}^2$-realization but its absence in the $\mathbb{C}^1$-realization of RST. Let us remark also that the *conservation equations* (6.32b) and (6.32c) receive the status of *conservation laws* only for vanishing field strength (i.e., $F_{\mu\nu} \equiv 0$) because in this case the potential A_μ can be gauged off ($A_\mu \Rightarrow 0$) and consequently the covariant derivative D_μ can be replaced by ∇_μ. We shall make use of this fact below when considering the Zitterbewegung over an FRW universe.

6.3. Density Dynamics

The results (4.11)–(4.12) for the energy density U and pressure P show that these quantities will be composed of both Hamiltonian coefficients and physical densities collected into the intensity matrix $\mathcal{I}$ [i.e., Eq. (6.15) for the $\mathbb{R}^2$-realization]. Therefore, in order to discuss the Zitterbewegung as it emerges from the solutions to the Einsteinian system (4.8), it finally becomes necessary to consider the dynamical equations for the physical densities. In compact form, the desired density dynamics has already been written down as the relativistic von Neumann equation (3.9), so that we merely have to specify this equation for the physical densities emerging in the decomposition of $\mathcal{I}$, (6.15). This procedure yields for the external density ρ, (6.10), as the gauge-invariant part of $\mathcal{I}$

$$\partial_\mu \rho = 2\rho \cdot L_\mu + q \cdot l_\mu + 4s \cdot N_\mu \qquad (6.33)$$

Observe again that the derivative of the invariant ρ contains a contribution of the internal motion (the last two terms), albeit in a gauge-invariant combination. Similarly, the field equations for the internal densities q and s are found from Eq. (3.9) as

$$D_\mu q = 4s \cdot K_\mu + \rho \cdot l_\mu + 2q \cdot L_\mu \qquad (6.34a)$$

$$D_\mu s = -q \cdot K_\mu + \rho \cdot N_\mu + 2s \cdot L_\mu \qquad (6.34b)$$

(Hint: A quick check of this result consists in its correct gauge covariance.)

Now, there is an interesting point with the present density dynamics which has been already observed in connection with the 2-particle systems [16], namely the link between the amplitude field $L(x)$ and the physical densities. It has already been remarked during the discussion of the current ${}^{(2)}j_{\mu}$, (6.28), that for the Klein–Gordon theory one must identify L^2 and ρ. But this identification must obviously be generalized for the present $\mathbb{R}^2$-realization of RST. Indeed, it is suggestive to connect the physical densities with the amplitude field $L(x)$ via some "renormalization factor" Z, i.e., we put

$$\rho = Z_I \cdot L^2 \tag{6.35a}$$

$$q = Z_D \cdot L^2 \tag{6.35b}$$

$$s = Z_P \cdot L^2 \tag{6.35c}$$

Since the amplitude field $L(x)$ is an $SO(2)$ gauge invariant, the renormalization factors inherit their transformation behavior from the corresponding physical densities, i.e.,

$$Z_I' \equiv Z_I \tag{6.36a}$$

$$Z_D' = Z_D \cdot \cos 2a + 2Z_P \cdot \sin 2a \tag{6.36b}$$

$$2Z_P' = 2Z_P \cdot \cos 2a - Z_D \cdot \sin 2a \tag{6.36c}$$

Furthermore, the field equations for the physical densities (6.33)–(6.34) are converted to the corresponding equations for the renormalization factors

$$\partial_{\mu} Z_I = Z_D \cdot l_{\mu} + 4Z_P \cdot N_{\mu} \tag{6.37a}$$

$$D_{\mu} Z_D = 4Z_P \cdot K_{\mu} + Z_I \cdot l_{\mu} \tag{6.37b}$$

$$D_{\mu} Z_P = -Z_D \cdot K_{\mu} + Z_I \cdot N_{\mu} \tag{6.37c}$$

where the covariant derivatives have to be defined in an appropriate way [cf. (6.17)]:

$$D_{\mu} Z_D \doteqdot \partial_{\mu} Z_D + 4A_{\mu} \cdot Z_P \tag{6.38a}$$

$$D_{\mu} Z_P \doteqdot \partial_{\mu} Z_P - A_{\mu} \cdot Z_D \tag{6.38b}$$

The interesting point with the renormalization factors is that they give rise to the introduction of the desired trembling variable describing the Zitter degree of freedom. This is readily recognized by a closer inspection of the *Fierz deviation* Δ_F [24],

$$\Delta_F \doteqdot (\mathrm{tr}\, \mathcal{I})^2 - \mathrm{tr}(\mathcal{I}^2) = \tfrac{1}{2}(\rho^2 - q^2 - (2s)^2) \tag{6.39}$$

which vanishes for the pure states; see the discussion of the Fierz identity (6.18). One can easily show by means of the density dynamics (6.33)–(6.34) that the Fierz deviation Δ_F obeys the field equation

$$\partial_\mu \Delta_F = 4\Delta_F \cdot L_\mu \tag{6.40}$$

and therefore can be expressed in terms of the amplitude field $L(x)$ as

$$\Delta_F(x) = \Delta_{F,in} \cdot \left(\frac{L(x)}{L_{in}}\right)^4 \tag{6.41}$$

$$(\Delta_{f,in}, L_{in} \mid = \text{const})$$

If this result is resubstituted into Eq. (6.39) with the simultaneous elimination of the physical densities in favor of the renormalization factors, we get the following constraint:

$$Z_I^2 - Z_D^2 - (2Z_P)^2 = \text{const} \tag{6.42}$$

Here it is obvious that there are three essentially different cases according to whether the constant on the right of (6.42) is positive, zero, or negative. Without loss of generality we can put

$$Z_I^2 - Z_D^2 - (2Z_P)^2 = \sigma_* \tag{6.43}$$

where the constant σ_* adopts the values 0, ± 1 and therefore plays a part quite analogous to the topological index σ for the Einsteinian system (4.8). Introducing the gauge invariant Z_{II} through

$$Z_{II} = \sqrt{Z_D^2 + (2Z_P)^2} \tag{6.44}$$

we have from (6.43)

$$Z_I^2 - Z_{II}^2 = \sigma_* \tag{6.45}$$

and therefore we can parametrize the *positive mixtures* ($\sigma_* = +1$) by means of an internal variable ζ

$$Z_I = \pm \cosh \zeta \tag{6.45a}$$

$$Z_{II} = \sinh \zeta \tag{6.45b}$$

the *negative mixtures* ($\sigma_* = -1$) by

$$Z_I = \sinh \zeta \tag{6.46a}$$

$$Z_{II} = \pm \cosh \zeta \tag{6.46b}$$

and finally the *pure states* ($\sigma_* = 0$) by

$$Z_I = Z_{II} = e^\zeta \tag{6.47}$$

Furthermore, for any one of the three situations we can introduce an *angular variable* η through

$$Z_D = Z_{II} \cdot \cos\eta \tag{6.48a}$$

$$2Z_P = Z_{II} \cdot \sin\eta \tag{6.48b}$$

and this object $\eta(x)$ must then change under the $SO(2)$ gauge transformations (6.8) according to

$$\eta \Rightarrow \eta' = \eta - 2a \tag{6.49}$$

in order that the renormalization factors Z_D and Z_P, (6.48), obey the former transformation law (6.36). Thus the pure states ($\sigma_* = 0$) are seen to sweep out the "Fierz cone" in density configuration space, whereas the positive mixtures ($\sigma_* = +1$) occupy the two-part hyperboloid within the Fierz cone and the negative mixtures ($\sigma_* = -1$) are characterized by the unparted hyperboloid outside the Fierz cone (Fig. 5).

Once the internal variable ζ and the angular variable η have been introduced, one would like to see also their equations of motion [to be deduced from the renormalization dynamics (6.37)]. To this end, it is very convenient to combine first the localization coefficients l_μ and N_μ, (6.21), into two *gauge-invariant* combinations g_μ and h_μ:

$$g_\mu = \sin\eta \cdot l_\mu - \cos\eta \cdot (2N_\mu) \tag{6.50a}$$

$$h_\mu = \cos\eta \cdot l_\mu + \sin\eta \cdot (2N_\mu) \tag{6.50b}$$

and then the field equations for ζ and η read for all three cases ($\sigma_* = 0, \pm 1$)

$$\partial_\mu \zeta = h_\mu \tag{6.51a}$$

$$\eta_\mu = -2\left(K_\mu + \frac{Z_I}{2Z_{II}}\, g_\mu\right) \tag{6.51b}$$

The remarkable point here is again the question of gauge covariance: since the angular variable η is itself not a gauge invariant [cf. (6.49)] its partial derivative ($\partial_\mu \eta$) must first be combined with the gauge potential A_μ into a *gauge-invariant* object η_μ,

$$\eta_\mu \doteqdot \partial_\mu \eta - 2A_\mu \tag{6.52}$$

before it can enter the invariant field equation (6.51b)!

Finally, it is also instructive to convince oneself of the internal character of the variable ζ. For this purpose, rewrite the internal current ${}^{(i)}j_\mu$, (6.29), in terms of the newly introduced objects and find, e.g., for the positive mixture ($\sigma_* = +1$)

$${}^{(i)}j_\mu = \frac{\hbar}{Mc}\left(K_\mu \cdot (\cosh\zeta - 1) + \frac{1}{2}\, g_\mu \cdot \sinh\zeta\right) \cdot L^2 \tag{6.53}$$

Thus, the internal current must vanish when the internal variable ζ approaches

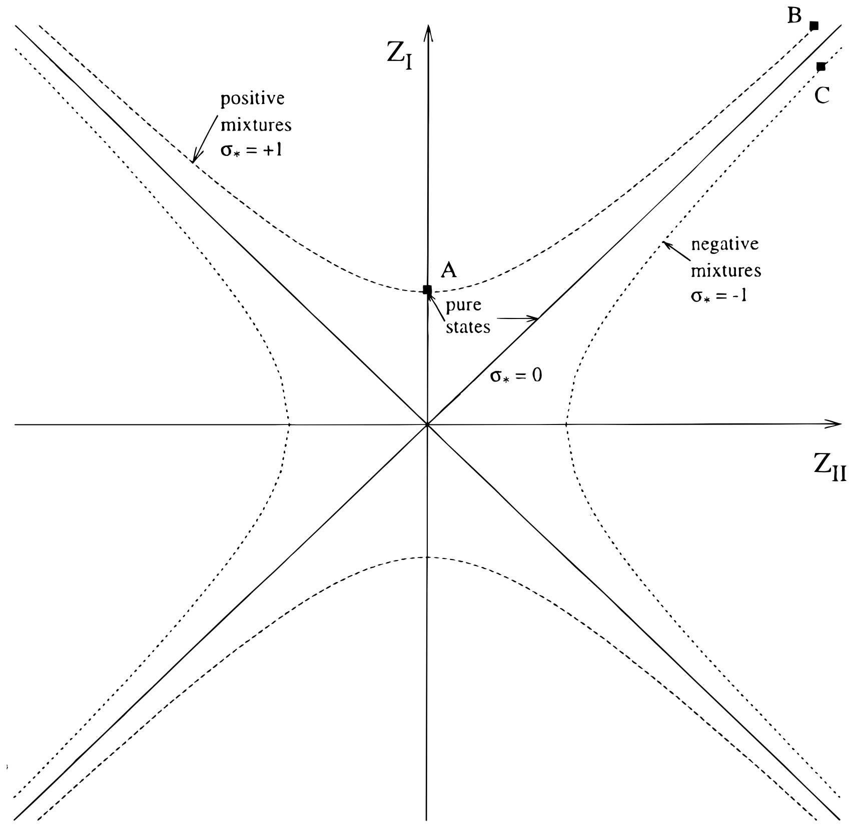

Fig. 5. Mixtures and pure states The relativistic von Neumann equation (3.9) divides the densitiy configuration space into three dynamically disconnected subsets: mixtures of positive and negative type and pure states which sweep out the Fierz cone ($\sigma_* = 0$). The positive mixtures approach a pure state for $\zeta \to 0$ (A) and $\zeta \to \infty$ (B), whereas the negative mixtures can approach the pure states *only* for $\zeta \to \infty$ (C). A "quantum jump" is a sudden transition from a pure state (A) to another pure state (B) where the intermediate configurations are quasi-pure states, i.e., a special subset of the positive mixtures.

its trivial value ($\zeta \to 0$). As a consequence, both currents ${}^{(1)}j_\mu$, (6.27b), and ${}^{(2)}j_\mu$, (6.28), become identical, which signals that the internal degree of freedom has disappeared from the dynamics and we are left with the conventional Klein–Gordon theory, [cf. the Klein–Gordon current j_μ, (5.9)]. This situation suggests we introduce a new type of field configuration: the "quasi-pure state." Here the conserved internal charge z_*, (6.31), does vanish, i.e., $z_* = 0$, but the corresponding internal current ${}^{(i)}j_\mu$, (6.53), may (or may not) be different from zero. Obviously this is a slight generalization of the situation where the internal current ${}^{(i)}j_\mu$, (6.53), itself vanishes [i.e., $\zeta(x) \equiv 0$]; equivalently, where the RST current ${}^{(2)}j_\mu$, (6.28), coincides with its Klein–Gordon counterpart ${}^{(1)}j_\mu$, (6.27b). For later purposes, we remark that the pure states can be approached not only for $\zeta \to 0$, but also for $\zeta \to \infty$.

6.4. Hamiltonian Dynamics Revisited

As the preceding arguments have shown, the density dynamics could be most conveniently expressed in terms of the newly introduced vector fields g_μ and h_μ [cf. (6.50)]. However, in order to have a closed dynamical system, we should now also express the former Hamiltonian dynamics in terms of those vectors g_μ and h_μ. But apart from this formal viewpoint, the use of g_μ and h_μ (in place of N_μ and l_μ) will provide us with further insight into the RST.

First, observe that the amplitude equation (6.32a) reads in terms of the new vectors g_μ and h_μ as follows:

$$\Box L + \left\{ \left(\frac{Mc}{\hbar}\right)^2 - K^\mu K_\mu + \frac{g^\mu g_\mu + h^\mu h_\mu}{4} \right\} \cdot L = 0 \tag{6.54}$$

Clearly, one is tempted to consider this equation as the immediate generalization of the Klein–Gordon case (5.32) where the vectors g_μ and h_μ describe here the additional internal degree of freedom which is missing in the Klein–Gordon theory. However, things are not so simple and the reason for this is that the present kinetic field K_μ does not have the field strength $F_{\mu\nu}$ as its curl [cf. (6.26)], which, however, is the case for the Klein–Gordon analogue (5.15). Therefore, before we can safely estimate the effects of the internal motion inherent in the $\mathbb{R}^2$-realization, we first have to understand better the role played by the kinetic field K_μ within both realizations of RST. This insight can be attained by a closer inspection of the curl relations for the new vectors g_μ and h_μ.

A modified kinetic field $'K_\mu$ which has the field strength $F_{\mu\nu}$ as its curl

$$\nabla_\mu {'K_\nu} - \nabla_\nu {'K_\mu} = F_{\mu\nu} \tag{6.55}$$

is readily available by simply observing the fact that the gauge-invariant object η_μ, (6.52), has the desired field strength $F_{\mu\nu}$ as its curl, i.e.,

$$\nabla_\mu \eta_\nu - \nabla_\nu \eta_\mu = -2F_{\mu\nu} \tag{6.56}$$

Therefore, remembering the former relationship (6.51b) between K_μ and η_μ, we immediately find the desired field $'K_\mu$, (6.55), as

$$'K_\mu \equiv -\frac{1}{2}\,\eta_\mu = K_\mu + \frac{Z_I}{2Z_{II}}\,g_\mu \tag{6.57}$$

This conclusion says that both kinetic fields K_μ and $'K_\mu$ differ by some vector field G_μ (“exchange vector field”; see ref. 16)

$$'K_\mu = K_\mu - G_\mu \tag{6.58}$$

the curl of which is found from a combination of both curl relations (6.26) and (6.55) as

$$\begin{aligned} \nabla_\mu G_\nu - \nabla_\nu G_\mu &= l_\mu N_\nu - l_\nu N_\mu \\ &\equiv \frac{1}{2}(g_\mu h_\nu - g_\nu h_\mu) \doteqdot G_{\mu\nu} \end{aligned} \tag{6.59}$$

The "exchange field strength" $G_{\mu\nu}$ introduced in this way is gauge invariant as is the exchange vector G_μ, which is read off directly from Eq. (6.57)–(6.58) as

$$G_\mu = -\frac{1}{2}\frac{Z_I}{Z_{II}}\, g_\mu \tag{6.60}$$

Perhaps this result (6.60) for G_μ may appear to have been obtained by a somewhat indirect approach, but its consistency check is a nice exercise:

(i) Compute the left-hand side of Eq. (6.59) by reference to the solution for G_μ, (6.60).

(ii) Thereby observe the derivatives of the renormalization factors Z_I, Z_{II} [to be in agreement with their constraint (6.45)]

$$\partial_\mu Z_I = Z_{II} \cdot h_\mu \tag{6.61a}$$

$$\partial_\mu Z_{II} = Z_I \cdot h_\mu \tag{6.61b}$$

(iii) Furthermore, use the curl relations for g_μ and h_μ [to be deduced from the former curl relations for l_μ and N_μ, (6.23)]

$$\nabla_\mu g_\nu - \nabla_\nu g_\mu = \frac{Z_I}{Z_{II}}(h_\mu g_\nu - h_\nu g_\mu) \tag{6.62a}$$

$$\nabla_\mu h_\nu - \nabla_\nu h_\mu = 0 \tag{6.62b}$$

and then find the result identical to the right-hand side of Eq. (6.59).

After having thus gained some confidence in the new form of the Hamiltonian dynamics, one can trace back the new vectors g_μ and h_μ, (6.50), to certain scalar fields (ζ and χ, say). Of course the curl relation (6.62b) immediately identifies the vector h_μ as a gradient field, but this was already known through (6.51a). What is new is that the curl relation for g_μ, (6.62a), gives rise to a scalar field (χ, say) because the general solution of (6.62a) for g_μ is given by

$$g_\mu = Z_{II} \cdot \partial_\mu \chi \tag{6.63}$$

Both scalar fields ζ and χ will be of considerable help for discussing the effects of the internal motion upon the Zitterbewegung in the next section.

To conclude the Hamiltonian dynamics in its new form, let us mention also the source equations for the new vectors g_μ and h_μ, to be deduced from the analogous source equations (6.32b)–(6.32c) for the old vectors l_μ, and N_μ:

$$\nabla^\mu(L^2 g_\mu) = -2L^2 \cdot ({}'K_\mu \cdot h^\mu) \tag{6.64a}$$

$$\nabla^\mu(L^2 h_\mu) = 2L^2 \cdot ({}'K_\mu \cdot g^\mu) \tag{6.64b}$$

Since both g_μ and h_μ can now be traced back to the scalars χ and ζ, the system (6.64) is effectively a coupled second-order system for those scalar fields. Let us mention also that the source equations (6.64) are needed when one wants to verify the general conservation law (3.4) for the RST current ${}^{(2)}j_\mu$, (6.28), which reads in terms of the new variables

$${}^{(2)}j_\mu = \frac{\hbar}{Mc}\left\{Z_I \cdot K_\mu + \frac{1}{2} Z_{II} \cdot g_\mu\right\} \cdot L^2 \tag{6.65}$$

e.g., for the positive mixtures (6.45)

$${}^{(2)}j_\mu \Rightarrow \frac{\hbar}{Mc}\left\{\cosh\zeta \cdot K_\mu + \frac{1}{2}\sinh\zeta \cdot g_\mu\right\} \cdot L^2 \tag{6.66}$$

Observe here again the striking similarity with the Dirac currrent ${}^{(D)}j_\mu$, (2.15), which yields the motivation to introduce the internal current ${}^{(i)}j_\mu$, (6.53), as the RST counterpart of the polarization current ${}^{(P)}j_\mu$, (2.20b).

The vanishing of the internal current ${}^{(i)}j_\mu$ for $\zeta \to 0$ is accompanied by a similar effect for the new amplitude equation (6.54), which reads in terms of the scalar fields χ and ζ

$$\Box L + \left\{\left(\frac{Mc}{\hbar}\right)^2 - K^\mu K_\mu + \frac{Z_{II}^2 \partial^\mu\chi \cdot \partial_\mu\chi + \partial^\mu\zeta \cdot \partial_\mu\zeta}{4}\right\} \cdot L = 0 \tag{6.67}$$

Indeed for the positive mixtures (6.45), i.e., $Z_{II} \Rightarrow \sinh\zeta$, we recover again the Klein–Gordon case (5.32) in the limit $\zeta \to 0$. But why do we not regain the Klein–Gordon situation for the *negative* mixtures in the limit $\zeta \to 0$ when the internal motion is thought to come to rest?

6.5. Approaching the Pure States

The fact that, up to now, we can approach the pure states only from the region of positive mixtures ($\zeta \to 0$), but not from that of negative mixtures (see Fig. 5), brings us to consider more thoroughly the pure-state limit. In view of the pseudo-Euclidian geometry of the density configuration space, one is strongly reminded of the analogous situation in special relativity where the world line of some point particle, if accelerated up to the speed of light, approaches more and more the *light cone*. Analogously we would suppose for our present situation that the pure states, being represented by the *Fierz cone* in density configuration space, should be approached by mixtures of both kinds for $\zeta \rightarrow \infty$ (not only for $\zeta \rightarrow 0$, as considered up to now for the positive mixtures). Indeed, a simple geometric argument demonstrates that both types of hyperboloids approach the Fierz cone for $\zeta \rightarrow \infty$ (see Fig. 5). We are going now to recast this intuitive geometric idea into rigorous analytical results.

In order to make manifest the Klein–Gordon theory as the limiting case ($\zeta \rightarrow \infty$) for both mixtures, we first introduce a *modified localization field* ${}'L_\mu$, quite analogously to the case of the kinetic field ${}'K_\mu$, (6.57):

$$ {}'L_\mu \doteqdot L_\mu + \frac{1}{2}\frac{Z_I}{Z_{II}}\, h_\mu \equiv \frac{\partial_\mu {}'L}{{}'L} \tag{6.68} $$

Clearly such a construction gives rise to the emergence of a *modified amplitude field* ${}'L(x)$,

$$ {}'L(x) \doteqdot \sqrt{Z_{II}} \cdot L(x) \tag{6.69} $$

The use of these modified fields puts now the results for the mixtures in a form which is closely related to the Klein–Gordon results and thus elucidates their limiting character.

First, consider the RST current ${}^{(2)}j_\mu$, (6.65), which reads now in terms of the modified fields

$$ {}^{(2)}j_\mu = \frac{\hbar}{Mc} \cdot \left\{ \frac{Z_I}{Z_{II}} \cdot {}'K_\mu - \frac{1}{2}\frac{\sigma_*}{Z_{II}^2} \cdot g_\mu \right\} \cdot {}'L^2 \tag{6.70} $$

$$ (\nabla^{\mu\,(2)} j_\mu = 0) $$

For the *pure states* (6.47), where $\sigma_* = 0$, this current is simplified to

$$ {}^{(2)}j_\mu \Rightarrow \frac{\hbar}{Mc}\, {}'K_\mu\, {}'L^2 \tag{6.71} $$

which is just the Klein–Gordon result (5.9) via the identifications

$$ \text{RST} \quad \left\{ \begin{matrix} {}'K_\mu \Leftrightarrow K_\mu \\ {}'L_\mu \Leftrightarrow L_\mu \end{matrix} \right\} \quad \text{KGT} \tag{6.72} $$

However, the point here is that the Klein–Gordon result (6.71) arises also for *both mixtures* ($\sigma_* = \pm 1$) in the limit case $\zeta \rightarrow \infty$, which accurately meets with our expectations mentioned above.

For the amplitude equation, the situation is not quite so clear. Reformulating Eq. (6.67) in terms of the modified fields yields

$$\Box' L + \left\{ \left(\frac{Mc}{\hbar} \right)^2 - 'K^\mu \cdot 'K_\mu - \frac{1}{4} \sigma_* \left[(\partial^\mu \chi)(\partial_\mu \chi) - \left(\frac{1}{Z_{II}} \right)^2 (\partial^\mu \zeta)(\partial_\mu \zeta) \right] \right\} \cdot 'L = 0 \tag{6.73}$$

Of course, for the pure states ($\sigma_* = 0$) we again arrive at the Klein–Gordon amplitude equation (5.32) via the former identifications (6.72). But for the mixtures ($\sigma_* = \pm 1$) we need an extra argument for the limiting case ($\zeta \to \infty$), which we will supply below when discussing the Zitterbewegung. For that purpose we shall also need the modified forms of the source equations (6.64), which are rewritten as

$$\nabla^\mu \left(\frac{'L^2 \cdot \partial_\mu \zeta}{Z_{II}} \right) = 2\, 'L^2 \cdot ('K^\mu \cdot \partial_\mu \chi) \tag{6.74a}$$

$$\nabla^\mu \, ('L^2 \cdot \partial_\mu \chi) = -2\, \frac{'L^2}{Z_{II}} \cdot ('K^\mu \cdot \partial_\mu \zeta) \tag{6.74b}$$

Finally, the discussion of Zitterbewegung will also require the specification of the energy-momentum density $T_{\mu\nu}$, (3.10b). Following that constructive receipe for $T_{\mu\nu}$ with the energy-momentum operator $\mathcal{T}_{\mu\nu}$ being given by (3.21) yields $T_{\mu\nu}$ as a sum of three contributions:

$$T_{\mu\nu} = \frac{Z_I}{Z_{II}} \cdot {}^{(KG)}T_{\mu\nu} + {}^{(m)}T_{\mu\nu} + {}^{(i)}T_{\mu\nu} \tag{6.75}$$

Here the first part ${}^{(KG)}T_{\mu\nu}$ coincides with the Klein–Gordon result (albeit expressed in terms of the modified fields)

$${}^{(KG)}T_{\mu\nu} = \frac{\hbar^2}{M} \left\{ 'K_\mu \, 'K_\nu + 'L_\mu \, 'L_\nu - \frac{1}{2} g_{\mu\nu} \cdot \left('K^\lambda \, 'K_\lambda + 'L^\lambda \, 'L_\lambda - \left(\frac{Mc}{\hbar} \right)^2 \right) \right\} \cdot 'L^2 \tag{6.76}$$

Clearly this part ${}^{(KG)}T_{\mu\nu}$, surviving the limit $\zeta \to \infty$, is due to the *external* motion and therefore is present already in the Klein–Gordon theory as a point-particle theory. But in addition to this now we have also an *internal* energy-momentum ${}^{(i)}T_{\mu\nu}$,

$${}^{(i)}T_{\mu\nu} = \sigma_* \cdot \frac{\hbar^2}{4M} \cdot \frac{Z_I}{Z_{II}^3} \cdot 'L^2 \cdot i_{\mu\nu} \tag{6.77a}$$

with the internal tensor $i_{\mu\nu}$ being given by

$$i_{\mu\nu} = g_\mu g_\nu + h_\mu h_\nu - \frac{1}{2} g_{\mu\nu}(g^\lambda g_\lambda + h^\lambda h_\lambda) \tag{6.77b}$$

This internal part ${}^{(i)}T_{\mu\nu}$ is completely missing for the pure states ($\sigma_* = 0$), but it does not vanish formally for $\zeta \to \infty$. Therefore this part requires a seperate discussion. The last term ${}^{(m)}T_{\mu\nu}$ is a mixed term and therefore describes the interaction between the external and internal motion:

$${}^{(m)}T_{\mu\nu} = -\sigma_* \cdot \frac{\hbar^2}{2M} \frac{{}'L^2}{{Z_{II}}^2} \cdot m_{\mu\nu} \tag{6.78a}$$

with the mixed tensor $m_{\mu\nu}$ being specified as

$$\begin{aligned} m_{\mu\nu} = g_\mu \cdot {}'K_\nu + g_\nu \cdot {}'K_\mu + h_\mu \cdot {}'L_\nu + h_\nu \cdot {}'L_\mu \\ - g_{\mu\nu}(g^\lambda \cdot {}'K_\lambda + h^\lambda \cdot {}'L_\lambda) \end{aligned} \tag{6.78b}$$

It is easy to see that this mixed part ${}^{(m)}T_{\mu\nu}$ actually vanishes not only exactly for the pure states ($\sigma_* = 0$), but also for the limiting case $\zeta \to \infty$ for both kinds of mixtures ($\sigma_* = \pm 1$).

7. OSCILLATIONS AND JUMPS

With the preceding preparations we arrive now at the right point of departure for studying the internal Zitterbewegung. Remember that for the $\mathbb{R}^2$-realization the complete dynamical system consists of the equations of motion for the *external* variables ${}'K_\mu$ and ${}'L$ and for the *internal* variables ζ and χ. Concerning the external variables, the dynamical system for the kinetic field ${}'K_\mu$ consists of the source equation (6.27) and the curl equation (6.55), whereas the amplitude field ${}'L(x)$ has to obey the wave equation (6.73). For the internal variables ζ and χ we have found the equations of motion (6.74). Since obviously both subsets of the dynamical equations are coupled to each other, one expects a distinct influence of the Zitterbewegung upon the external motion and this in turn must leave its imprint upon the space-time geometry via the Einstein equations (4.8). Here, the cosmological principle again simplifies the situation considerably and we shall now first reduce the general equations of motion due to the Robertson–Walker symmetry.

7.1. Internal Dynamics

The RW symmetry makes both the scalars ζ and χ depend exclusively upon the cosmic time Θ and this simplifies the corresponding dynamical system (6.74) into the following form:

$$\frac{d}{d\Theta}\left(\frac{\mathcal{R}^3\ {}'L^2\ \dot{\zeta}}{Z_{II}}\right) = 2'K \cdot (\mathcal{R}^3\ {}'L^2\ \dot{\chi}) \tag{7.1a}$$

$$\frac{d}{d\Theta}(\mathcal{R}^3\ {}'L^2\ \dot{\chi}) = -2'K \cdot \left(\frac{\mathcal{R}^3\ {}'L^2\ \dot{\zeta}}{Z_{II}}\right) \tag{7.1b}$$

Clearly such a result suggests we resort to a reparametrization by introducing an angular variable X of the following kind:

$$\frac{\mathcal{R}^3\ {}'L^2\ \dot{\zeta}}{Z_{II}} = \lambda_* \sin X \tag{7.2a}$$

$$\mathcal{R}^3\ {}'L^2\ \dot{\chi} = \lambda_* \cos X \tag{7.2b}$$

$$(\lambda_* = \text{const})$$

The equation of motion for the new variable X is then readily deduced from the preceding system (7.1) as

$$\dot{X} = 2\ {}'K \tag{7.3}$$

As an application of this reparametrization arrangement one can consider the amplitude equation (6.73), which now reappears in its RW-symmetric form as

$${}'\ddot{L} + 3H\ {}'\dot{L} + \left\{\left(\frac{Mc}{\hbar}\right)^2 - {}'K^2 - \frac{1}{4}\,\sigma_*\,\lambda_*^2 \cdot \frac{\cos 2X}{\mathcal{R}^6\ {}'L^4}\right\} \cdot {}'L = 0 \tag{7.4}$$

Observe here that for the pure states ($\sigma_* = 0$) we are actually led back again to the Klein–Gordon case (5.33); cf. the identifications (6.72). But additionally the Klein–Gordon form of the amplitude equation also applies to the mixtures ($\sigma_* \neq 0$), namely when the universe's size becomes very large ($\mathcal{R} \to \infty$) or when the new variable $2X$ approaches $\pi/2(\text{mod } \pi)$ for the pure-state limit $\zeta \to \infty$.

7.2. Equation of State

Next, consider the Einsteinian system (4.8), which as it stands is not yet complete, but requires the specification of an equation of state. Such a link between the pressure P and energy density U is supplied here by specifying these quantities in terms of the external and internal variables. The former result (6.75) for the energy-momentum density $T_{\mu\nu}$ says that U and P should be composed of three constituents:

$$U = \frac{Z_I}{Z_{II}} \cdot U_{KG} + U_m + U_i \tag{7.5a}$$

$$P = \frac{Z_I}{Z_{II}} \cdot P_{KG} + P_m + P_i \tag{7.5b}$$

Of course, the Klein–Gordon part is here the same as previously, (5.26)–(5.27), apart from the transcription (6.72) of the corresponding fields:

$$U_{KG} = \frac{\hbar^2}{2M}\left\{{}'K^2 + {}'\Lambda^2 + \left(\frac{Mc}{\hbar}\right)^2\right\} \cdot {}'L^2 \tag{7.6a}$$

$$P_{KG} = \frac{\hbar^2}{2M}\left\{{}'K^2 + {}'\Lambda^2 - \left(\frac{Mc}{\hbar}\right)^2\right\} \cdot {}'L^2 \tag{7.6b}$$

Similarly, for the mixed contribution we have now

$$U_m = P_m = -\sigma_*\lambda_* \frac{\hbar}{2M} \cdot \frac{{}'K \cos X + {}'\Lambda \sin X}{Z_{II}\mathcal{R}^3} \tag{7.7}$$

and finally for the internal part

$$U_i = P_i = \sigma_*\lambda_*{}^2 \frac{\hbar^2}{8M}\frac{Z_I}{Z_{II}}\frac{1}{\mathcal{R}^6 \cdot {}'L^2} \tag{7.8}$$

The dynamical system would now be complete if one could specify also the equation of motion for the kinetic field $'K$. However this variable will be treated in a somewhat different way, namely by reference to the conservation laws.

7.3. Conservation Laws

The discussion of Zitterbewegung for the Klein–Gordon theory has been facilitated considerably by means of the conservation law (5.31). This provided us with the understanding of the dynamical behavior of the kinetic field K. In a similar way, both conservation laws (6.27) for ${}^{(1)}j_\mu$ and (6.70) for ${}^{(2)}j_\mu$ will play an important part in the present discussion of the RST Zitterbewegung. We shall benefit from the simultaneous existence of *two* conservation laws.

In order to see how this works, let us first reformulate the first current ${}^{(1)}j_\mu$, (6.27b), in terms of the modified fields:

$${}^{(1)}j_\mu = \frac{\hbar}{Mc}\frac{{}'L^2}{Z_{II}} \cdot \left({}'K_\mu - \frac{Z_I}{2Z_{II}}\, g_\mu\right) \tag{7.9}$$

By virtue of the cosmological principle, this result is recast into the form

$${}^{(1)}j_\mu \Rightarrow \frac{\hbar}{Mc}\frac{{}'L^2}{Z_{II}} \cdot \left({}'K - \frac{Z_I}{2}\,\dot{\chi}\right) b_\mu \tag{7.10}$$

On the other hand, any current j_μ obeying simultaneously the cosmological principle (5.28) and the conservation law $\nabla^\mu j_\mu = 0$, (5.29), must look like

$$ {}^{(1)}j_\mu = \frac{\hbar}{Mc} \frac{{}^{(1)}I_*}{\mathcal{R}^3} b_\mu \qquad {}^{(1)}I_* = \text{const}) \tag{7.11} $$

Thus comparing both forms (7.10) and (7.11) for ${}^{(1)}j_\mu$ yields the constraint

$$ Z_{II} \cdot {}^{(1)}I_* = \mathcal{R}^3 \, {}'L^2 \left({}'K - \frac{Z_I}{2} \dot{\chi} \right) \tag{7.12} $$

The same argument can be applied also to the RST current (${}^{(2)}j_\mu$, (6.70), and then yields an analogous constraint:

$$ Z_{II} \cdot {}^{(2)}I_* = \mathcal{R}^3 \, {}'L^2 \left(Z_I \, {}'K - \frac{1}{2} \sigma_* \dot{\chi} \right) \tag{7.13} $$

Now combine Eqs. (7.12) and (7.13) to eliminate the external variables ${}'K$ and ${}'L$ and then find the following constraint for the internal variables ζ, X:

$$ Z_{II} \cdot \cos X = i_2 - i_1 \cdot Z_I \qquad \left(i_a \doteqdot \frac{2 \cdot {}^{(a)}I_*}{\lambda_*} = \text{const}, \quad a = 1, 2 \right) \tag{7.14} $$

Furthermore, differentiating this result with respect to cosmic time Θ and using the equations of motion for ζ, (7.2a), and for X, (7.3), yields the RST counterpart of the Klein–Gordon conservation law (5.31):

$$ \mathcal{R}^3 \, {}'L^2 \, {}'K = \frac{{}^{(2)}I_* \cdot Z_I - \sigma_* \cdot {}^{(1)}I_*}{Z_{II}} \tag{7.15} $$

Indeed, for the pure states ($\sigma_* = 0$) we have $Z_I = Z_{II}$ [cf. (6.47)] and the present RST result (7.15) coincides *exactly* with the previous Klein–Gordon case (5.31). But this is also recovered *asymptotically* for a mixture ($\sigma_* \neq 0$) in the limit of a pure state ($\zeta \to \infty$). The algebraic equation (7.15) completes our dynamical system and can be used for the determination of ${}'K$ in place of solving the corresponding differential equation for the kinetic field. Obviously, the constant I_* on the right-hand side of the Klein–Gordon case (5.31) has now been substituted by the function of ζ defined by the right-hand side of the present conservation law (7.15). This is an interesting result because it implies that, in contrast to the Klein–Gordon case, the RST Zitterbewegung comes in several modes, which we are now going to inspect in detail.

The emergence of different modes of solutions originates from the constraint (7.14), which admits the following solutions for ζ as a function of the angular variable X:

$$e^{\zeta} = \frac{i_2 \pm \sqrt{i_2^2 \mp i_1^2 \pm \cos^2 X}}{i_1 + \cos X}$$

$$\begin{Bmatrix} \text{upper} \\ \text{lower} \end{Bmatrix} \text{ sign in the root term} \Leftrightarrow \begin{Bmatrix} \text{positive} \\ \text{negative} \end{Bmatrix} \text{ mixture} \qquad (7.16)$$

Here it is easy to see that for the positive mixtures ($\sigma_* = +1$) there are three different ranges of the parameters $\{i_1, i_2\}$ which admit solutions for the present dynamical system: the "open" ($i_2 > i_1 > 1$) and "closed" ($i_1 > i_2 > \sqrt{i_1^2 - 1}$) configurations and the limiting case between these two types, $i_2 = i_1$ (see Fig. 6). For all the other regions of the parameter space there

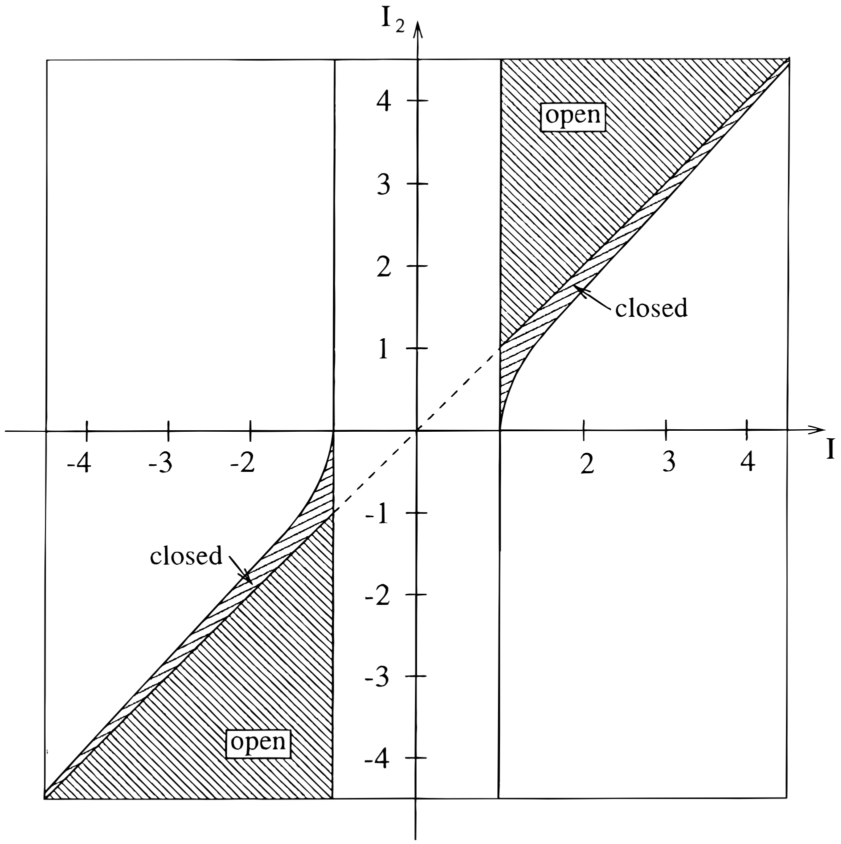

Fig. 6. Parameter space $\{i_1, i_2\}$. For the positive mixtures ($\sigma_* = +1$) there are three different ranges of the parameters i_1, i_2: open configurations ($i_2 > i_1 > 1$), limiting case ($i_2 = i_1$), and closed configurations $i_1 > i_2 > \sqrt{i_1^2 - 1}$. For the limiting case with $i_1 = i_2 < 1$ there occur jump solutions, whereas for $i_1 > 1$ the solutions are always of oscillatory character (i.e., Zitterbewegung).

do not exist solutions for our dynamical system. Correspondingly, the functional dependence of the internal variable ζ in terms of the angular variable X is an *open* curve extending to infinity $-\infty < X < +\infty$ (Fig. 7a), a *closed* curve (Fig. 7b), or, as the intermediate case, a closed curve running through infinity (Fig. 7c).

7.4. Zitterbewegung

Therefore with progression of cosmic time Θ, both the variables ζ and X are confined to a finite interval (Fig. 8b) for the closed case, whereas for

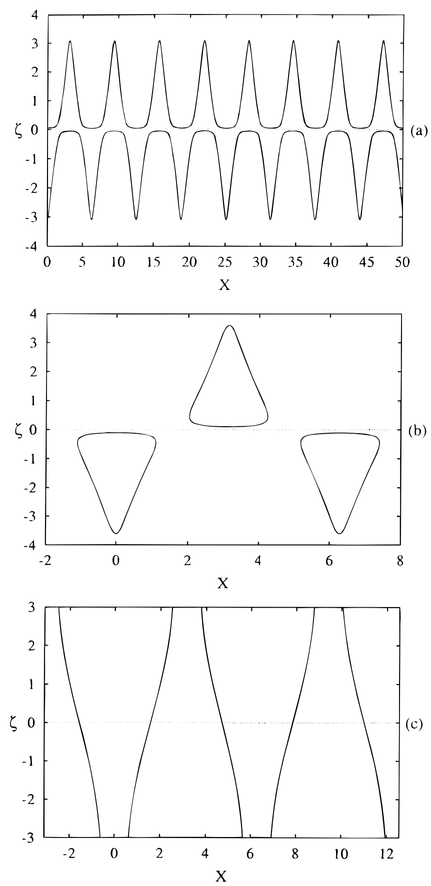

Fig. 7. Phase space of internal motion. Equation (7.16) describes three types of solutions for positive mixtures: open (a), closed (b) and the limiting case (c). The jump solutions are due to the limiting case with $i_1 < 1$ (c), whereas the closed and open cases lead to Zitterbewegung of oscillatory character.

the open case X can grow in an unrestricted way (Fig. 8a). For the intermediate case, the angle X remains finite, but ζ can extend up to infinity (Fig. 10). Since by their very definition both charges e_1 and e_2,

$$e_1 = \int_{S_3} {}^{(1)}j_\mu \, dS^\mu = 2\pi^2 \frac{\hbar}{Mc} \cdot {}^{(1)}I_* \tag{7.17a}$$

$$e_2 = \int_{S_3} {}^{(2)}j_\mu \, dS^\mu = 2\pi^2 \frac{\hbar}{Mc} \cdot {}^{(2)}I_* \tag{7.17b}$$

are found to be equal for this intermediate case (${}^{(1)}I_* = {}^{(2)}I_*$), which thus is identified as a quasi-pure state [cf. the discussion below Eq. (6.53)]. This different behavior of the phase angle X produces the corresponding distinc-

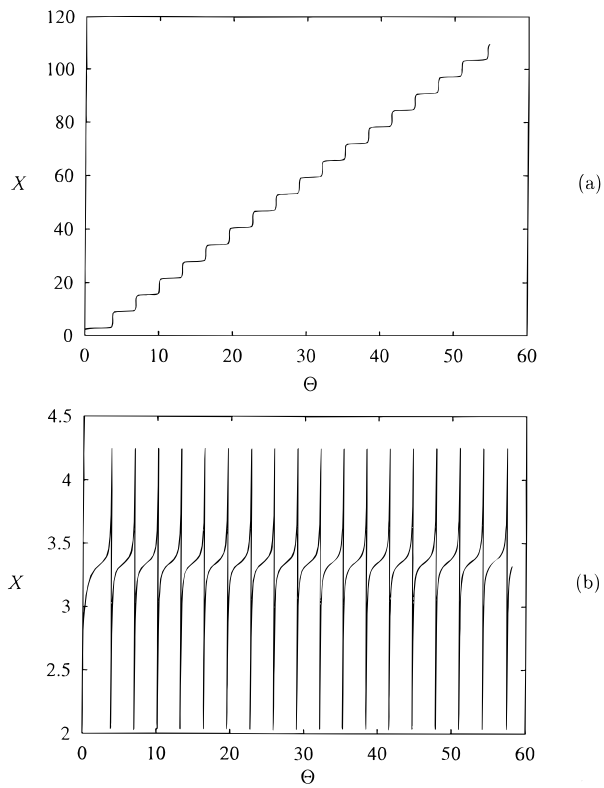

Fig. 8. Internal Zitterbewegung. For the open case (a), the angular variable X grows monotonically (roughly in steps of 2π); however, X is confined to a finite range for the closed case (b). In both cases, the internal variable ζ is subject to oscillatory motion, namely either by virtue of the monotonic growth of X for the *open* case (Fig. 7a), or by virtue of cycling around a closed curve in phase space for the *closed* case (Fig. 7b).

tions with respect to the pulse behavior of the kinetic field $'K$, (7.3): for the *open* case, the monotonic growth of X (Fig. 8a) implies that the kinetic pulses are always positive and of quantized strength ($\int {'K}\, d\Theta \approx 2\pi$), similar to the Klein–Gordon theory (Fig. 4); but for the *closed* case, the oscillatory behavior of X (Fig. 8b) must lead to a vanishing pulse strength: ($\int {'K}\, d\Theta = 0$), so that the values of $'K(\Theta)$ become both positive and negative during a pulse.

But from a more qualitative point of view, the pulse behavior of the kinetic field looks very similar to the Klein–Gordon case (Fig. 4) and therefore does not need to be reproduced here. The same is also true for the amplitude field $L(x)$ as a solution of the amplitude equation (7.4), which again looks very similar to the Klein–Gordon analogue (Fig. 3). Finally, let us mention that the closed universe recollapses for the mixtures in the same way as encountered for the pure states, (see the Klein–Gordon case; Fig. 1).

In summary, we can say that the internal degree of freedom in RST leads to a much richer structure of the Zitterbewegung in comparison to the point-particle case of the Klein–Gordon theory. But there occurs an even more striking phenomenon in RST, namely the jump solutions, which are of nonoscillatory character and therefore do not have any counterpart at all in Klein–Gordon theory.

7.5. Jumps

This completely new type of solution arises for a certain subset of the quasi-pure states, i.e., when the internal charge z_*, (6.31), vanishes. In genereal the vanishing of $z_* = e_2 - e_1$ does not imply the local coincidence of both currents ${}^{(1)}j_\mu$ and ${}^{(2)}j_\mu$, but for our homogenous and isotropic situation, both currents ${}^{(1)}j_\mu$ and ${}^{(2)}j_\mu$ must become identical just as a consequence of the vanishing of z_* [see the arguments leading to Eq. (7.11)]. In other words, for our highly symmetric situation (i.e., the RW symmetry) the vanishing of the internal charge z_* implies the vanishing of the internal current ${}^{(i)}j_\mu$ ($\equiv {}^{(2)}j_\mu - {}^{(1)}j_\mu$)! Observe, however, that this does not yet imply the “freezing” of the internal degree of freedom [i.e., $\zeta \Rightarrow 0$ in (6.53)].

On the other hand, not all of the quasi-pure states ($i_1 = i_2 =: i_*$) are jump solutions. Indeed, the general relation (7.16) is specified nontrivially for this type of solution to

$$e^{\zeta} = \frac{i_* - \cos X}{i_* + \cos X} \tag{7.18}$$

and this says that we must require $|i_*| < 1$ (Fig. 7c) in order that the two critical configurations with $\zeta = 0$ and $\zeta = \infty$ be admitted kinematically. But this restricts the range of the angular variable X to the interval $\pi/2 \leq X \leq$

$\cos^{-1} i_*$ (see Fig. 9); and thus one expects that the dynamical evolution will terminate at one or another of the critical field configurations with either $\zeta = 0$ or $\zeta = \infty$. Indeed this expectation is verified by the numerical integrations (Fig. 10). Additionally, one obtains here the result that the end configuration ($\zeta = \infty$) is reached after a *finite* time Θ_{end} depending upon the initial conditions. The occurence of a finite transition time is easily understandable by means of Eq. (7.2a): assume there that, in contrast to the internal variable ζ, the other variables $\mathcal{R}$, $'L$, X are changing slowly and then find the approximate equation for the rapidly changing ζ (positive mixtures):

$$\frac{\dot{\zeta}}{\sinh \zeta} \approx \text{const} \tag{7.19}$$

The asymptotic solution of this simple equation is

$$\zeta_{(\Theta \to \Theta_{\text{end}})} \approx -\ln(\Theta_{\text{end}} - \Theta) \tag{7.20}$$

$$(\Theta_{\text{end}} = \text{const})$$

which confirms the numerical result of a finite transition time Θ_{end}. Furthermore, observe that, when the initial configuration approaches the pure state with $\zeta = 0$ (initial points $3 \to 2 \to 1 \to \ldots$ in Fig. 9) the variable ζ tends to jump suddenly from its initial value ($\zeta = 0$) to the final value ($\zeta = \infty$).

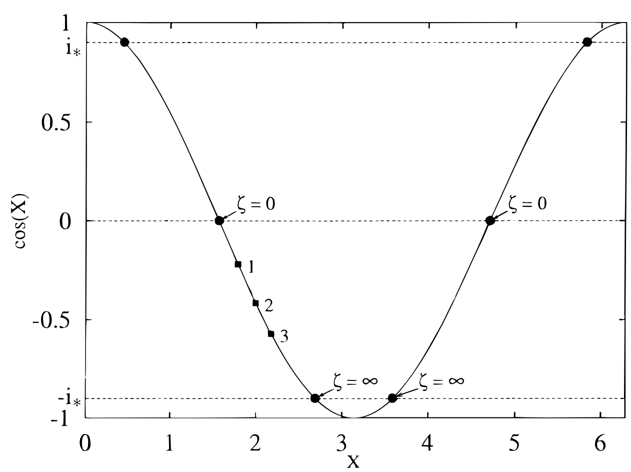

Fig. 9. Quasi-pure states. Jump solutions are obtained for the subset of quasi-pure states which have $i_1 \stackrel{!}{=} i_2 \doteqdot i_* < 1$ [cf. (7.14) and (7.17)]. All such solutions starting at points 1, 2, 3, ... terminate after a finite time at $\zeta = \infty$, which is a truly pure state. If one starts in the vicinity of $\zeta = 0$ (also a pure state), there occurs a sudden jump to the final pure state at $\zeta = \infty$ (Fig. 10). However the intermediate configurations are mixtures rather than pure states.

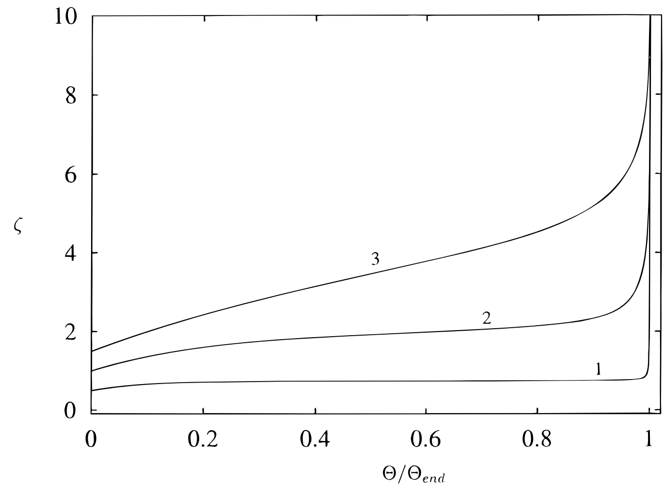

Fig. 10. Jump behavior of the internal variable ζ. If the initial configuration (marked by ■ in Fig. 9) approaches the pure state with $\zeta = 0$, the time behavior of ζ resembles more and more a jumplike transition to the final pure state with $\zeta = \infty$; cf. the deformation of solutions $3 \rightarrow 2 \rightarrow 1 \rightarrow \ldots$.

Such a solution obviously describes a jumplike transition form one pure state to another pure state through the space of quasi-pure states as a special subset of mixtures ($\rightsquigarrow$ "quantum jump"; Fig. 10).

The numerical integrations reveral also that the set of quasi-pure states is itself endowed with a certain structure (Fig. 11). If the initial configuration is close to the end configuration ($\zeta \rightarrow \infty$), e.g., 3 in Fig. 9, then the amplitude field ${}'L$ (Fig. 11a) and kinetic field ${}'K$ (Fig. 11b) adopt finite nonzero values at the endpoint ($\zeta = \infty$), which is in agreement with the Klein–Gordon conservation law (5.31) as the limit case (for $\zeta \rightarrow \infty$) of the RST conservation law (7.15). But when the initial configuration approaches the other pure state with $\zeta = 0$ (consider the sequence of initial conditions $3 \rightarrow 2 \rightarrow 1 \rightarrow \ldots$ in Fig. 9) then the final value of the amplitude field ${}'L$ tends to zero (Fig. 11a) and consequently the final value of the kinetic field ${}'K$ (Fig. 11b) must tend to infinity as required by the Klein–Gordon conservation law (5.31).

Finally, it is instructive to look at the energy density U and pressure P (Fig. 12). Perhaps it may appear as a surprise that these objects behave completely smoothly also for the jump solutions. Properly speaking, for a true quantum jump one should expect also a jumplike behavior for U and P. But for our present model, such a behavior is forbidden by the conservation laws. Remember that we have omitted the gauge interactions and thus

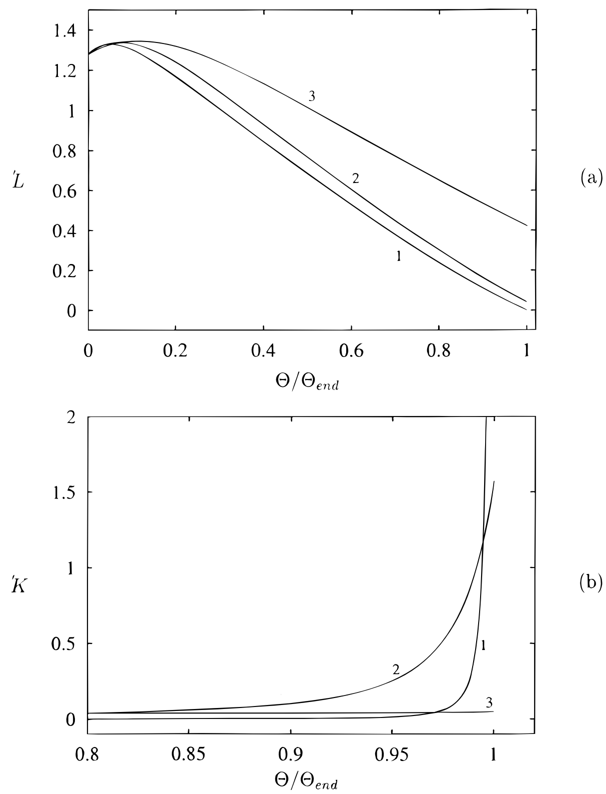

Fig. 11. Amplitude field $'L$ and kinetic field $'K$. For the limit case of a jump solution ($\{\zeta = 0\} \Rightarrow \{\zeta = \infty\}$) the final value of $'L$ becomes zero; see curve 1 in (a). The conservation laws (5.31) and (7.14) then demand that the final value of the kinetic field $'K$ must become infinite (b).

have restricted ourselves (but only for the sake of simplicity!) to the matter subsystem. But by virtue of the conservation laws, the matter subsystem must then be closed seperately [cf. (3.6)] which is also required in this case by the Einstein equations (4.1)($\rightsquigarrow$ Bianchi identity [23]). But with neglect of the gauge interactions we have no medium to carry away (or carry in) energy-momentum from (to) the matter subsystem (e.g., in the form of photons). Furthermore the RW symmetry, implied by the cosmological principle, forbids the emergence of gravitons as a medium for the transfer of energy-momentum to/from the matter subsystem. Thus there remains only the mechanism of the work-energy theorem in order to change the energy content of our symmetric matter arrangement (see Fig. 2). Therefore, in view of such restricted possibilities of energy exchange for

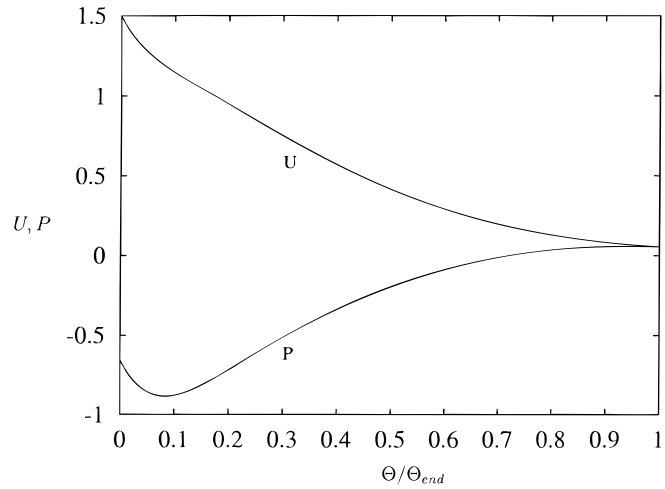

Fig. 12. Energy density *U* and pressure *P*. Since in our simplified model there is no medium for *transferring* energy-momentum from/to the matter system, the external quantities *U* and *P* behave strikingly smoothly during a jump process.

the matter subsystem, one should not be surprised about the relatively slight changes of *U* and *P* during a quantum jump (Fig. 12).

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